Grothendieck Conjecture for Hyperbolic Curves over Finitely Generated Fields of Positive Characteristic via Compatibility of Cyclotomes

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Abstract

Let p be a prime number. In the present paper, from the viewpoint of the compatibility/rigidity of group-theoretic cyclotomes, we revisit the anabelian Grothendieck Conjecture for hyperbolic curves over finitely generated fields of characteristic p established by A. Tamagawa, J. Stix, and S. Mochizuki. In particular, in light of this viewpoint, we give an alternative proof of the Grothendieck Conjecture for nonisotrivial hyperbolic curves over finitely generated fields of characteristic p obtained by J. Stix. In fact, by applying relatively recent results in anabelian geometry for hyperbolic curves over finite fields developed by M. Saïdi and A. Tamagawa, we discuss the J. Stix's result in a certain generalized situation, i.e., the geometrically pro-Σ setting, where Σ denotes the complement of a finite set of prime numbers that contains p in the set of all prime numbers. Moreover, by combining with a theorem in birational anabelian geometry obtained by F. Pop, we prove an absolute version of the geometrically pro-Σ Grothendieck Conjecture for nonisotrivial hyperbolic curves over the perfections of finitely generated fields of characteristic p. Let l be a prime number $\neq p$. In the present paper, we also investigate generalities concerning isotriviality, especially, we establish certain isotriviality criteria for hyperbolic curves with respect to both l-adic Galois representations and pro-l outer Galois representations.

Contents

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Introduction

Let p be a prime number. For each field F, we shall write $\overline{F}$ for the algebraic closure [determined up to isomorphisms] of F; F^{sep} ($\subseteq \overline{F}$) for the separable closure of F in $\overline{F}$; $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(F^{\mathrm{sep}}/F)$, i.e., the absolute Galois group of F. For each connected Noetherian scheme S, we shall write $\pi_1^{\text{ét}}(S)$ for the étale fundamental group of S, relative to a suitable choice of basepoint. For each hyperbolic curve S over a field [cf. Definition 1.1], we shall write $\pi_1^{\mathrm{tame}}(S)$ for the tame fundamental group of S, relative to a suitable choice of basepoint. Next, let F be a field; $F_0 \subseteq F$ a subfield; X an algebraic variety [i.e., separated, geometrically integral, and of finite type scheme] over F. Then we shall say that X is *isotrivial* relative to F_0 if $X \times_F \overline{F}$ descends to an algebraic variety over $\overline{F}_0$. Moreover, if F is of characteristic p, then we shall say that X is *isotrivial* if X is isotrivial relative to $\mathbb{F}_p$, i.e., the prime field of characteristic p.

In anabelian geometry, we often consider the geometricity/scheme-theoreticity of the isomorphisms [or more generally, homomorphisms] between various fundamental groups of "anabelian" algebraic varieties that include hyperbolic curves. This type of questions are originally posed by A. Grothendieck [cf. [5]] and referred to as Grothendieck Conjecture in this research area. The original question for hyperbolic curves over finitely generated fields in characteristic 0 was solved by A. Tamagawa (affine case — cf. [30], Theorem 0.4) and S. Mochizuki (proper case — cf. [15], Theorem A; [16], Theorem A). Moreover, in the characteristic 0 case, by using p-adic Hodge theory, S. Mochizuki proved a much more general statement for the hyperbolic curves over generalized sub-p-adic fields, i.e., subfields of finitely generated extension fields of the completion of the maximal unramified extension field of the field of p-adic numbers [cf. [16], Theorem A; [17], Theorem 4.12]. On the other hand, in the positive characteristic case, A. Tamagawa proved the Grothendieck Conjecture for affine hyperbolic curves over finite fields [cf. [30], Theorems 0.5, 0.6], and J. Stix proved the Grothendieck Conjecture for nonisotrivial [not necessarily, affine] hyperbolic curves over finitely generated transcendental extension fields of finite fields [cf. [28], Theorem 3.2; [29], Theorem 5.1.3]. Later, S. Mochizuki developed a highly nontrivial technique of cuspidalization and applied this technique to prove the Grothendieck Conjecture for arbitrary hyperbolic curves over finite fields [cf. [20], Theorem 3.12]. In summary, in the positive characteristic case, they proved the following results:

Theorem 0.1 (Tamagawa, Mochizuki). *Let k be a finite field; X_1, X_2 hyperbolic curves over k;*

$$\sigma : \pi_1^{\mathrm{tame}}(X_1) \xrightarrow{\sim} \pi_1^{\mathrm{tame}}(X_2)$$

a $\pi_1^{\mathrm{tame}}(X_2 \times_k \overline{k})$-outer isomorphism of profinite groups that lies over the identity automorphism of G_k. Then σ arises from a commutative diagram of schemes

$$\begin{array}{ccc} X_1 \times_k \overline{k} & \xrightarrow{\sim} & X_2 \times_k \overline{k} \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{\sim} & X_2, \end{array}$$

where the vertical arrows denote the natural projection morphisms; the horizontal arrows denote isomorphisms of schemes.

Theorem 0.2 (Stix). *Let k be a finitely generated transcendental extension field of $\mathbb{F}_p$; X_1 a nonisotrivial hyperbolic curve over k; X_2 a hyperbolic curve over k;*

$$\sigma : \pi_1^{\mathrm{tame}}(X_1) \xrightarrow{\sim} \pi_1^{\mathrm{tame}}(X_2)$$

a $\pi_1^{\mathrm{tame}}(X_2 \times_k \overline{k})$-outer isomorphism of profinite groups that lies over the identity automorphism of G_k. Then, after replacing X_1 or X_2 by a suitable Frobenius twist of them, σ arises from an isomorphism

$$X_1 \xrightarrow{\sim} X_2$$

over k. [Here, we note that the Frobenius twists do not change the tame fundamental groups.]

Next, write $\mathfrak{Primes}$ for the set of prime numbers. Let $\Sigma \subseteq \mathfrak{Primes}$ be a nonempty subset. More recently, for large Σ, M. Saïdi and A. Tamagawa proved a geometrically pro-Σ version of the Grothendieck Conjecture for hyperbolic curves over finite fields [cf. [26], Theorem 1; [27], Theorem D], which may be regarded as a refinement of Theorem 0.1. In order to state their results, we introduce some notations. For each profinite group G, we shall write G^{Σ} for the maximal pro-Σ quotient of G. For each algebraic variety S over a field F, we shall write

$$\Delta_S \stackrel{\mathrm{def}}{=} \pi_1^{\text{ét}}(S \times_F \overline{F})^{\Sigma}; \quad \Pi_S \stackrel{\mathrm{def}}{=} \pi_1^{\text{ét}}(S)/\mathrm{Ker}(\pi_1^{\text{ét}}(S \times_F \overline{F}) \twoheadrightarrow \Delta_S),$$

where $\pi_1^{\text{ét}}(S_{\overline{F}}) \twoheadrightarrow \Delta_S$ denotes the natural surjection. In particular, we have an exact sequence

$$1 \longrightarrow \Delta_S \longrightarrow \Pi_S \longrightarrow G_F \longrightarrow 1$$

of profinite groups. Then they proved the following result:

Theorem 0.3 (Saïdi-Tamagawa)**.** *Let k be a finite field; X_1, X_2 hyperbolic curves over k;*

$$\sigma : \Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}$$

a Δ_{X_2}-outer isomorphism of profinite groups that lies over the identity automorphism of G_k. Suppose that the complement $\mathfrak{Primes} \setminus \Sigma$ is finite and contains p [cf. Remark 2.6.1]. Then σ arises from a commutative diagram of schemes

$$\begin{array}{ccc} X_1 \times_k \overline{k} & \xrightarrow{\sim} & X_2 \times_k \overline{k} \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{\sim} & X_2, \end{array}$$

where the vertical arrows denote the natural projection morphisms; the horizontal arrows denote isomorphisms of schemes.

In the present paper, from the viewpoint of the compatibility/rigidity of group-theoretic cyclotomes associated to hyperbolic curves over fields of characteristic p [i.e., the duals of the second cohomology groups of the geometric pro-prime-to-p fundamental groups with coefficients in $\widehat{\mathbb{Z}}^{\mathfrak{Primes} \setminus \{p\}}$ in the case of proper hyperbolic curves — cf. Definition 2.3], we generalize Theorems 0.2, 0.3 as follows [cf. special cases of Theorems 4.5; 4.10]]:

Theorem A. *Let k be a finitely generated transcendental extension field of $\mathbb{F}_p$; X_1 a nonisotrivial hyperbolic curve over k; X_2 a hyperbolic curve over k;*

$$\sigma : \Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}$$

a Δ_{X_2}-outer isomorphism of profinite groups that lies over the identity automorphism of G_k. Suppose that the complement $\mathfrak{Primes} \setminus \Sigma$ is finite and contains p. Then, after replacing X_1 or X_2 by a suitable Frobenius twist of them, σ arises from an isomorphism

$$X_1 \xrightarrow{\sim} X_2$$

over k. [Here, we note that the Frobenius twists do not change the étale fundamental groups.]

Theorem B. *Let k_1, k_2 be the perfections of finitely generated transcendental extension fields of $\mathbb{F}_p$; X_1 a nonisotrivial hyperbolic curve over k_1; X_2 a hyperbolic curve over k_2. Suppose that the complement $\mathfrak{Primes} \setminus \Sigma$ is finite and contains p. Then the natural map*

$$\mathrm{Isom}(X_1, X_2) \longrightarrow \mathrm{OutIsom}(\Pi_{X_1}, \Pi_{X_2})$$

is bijective, where

$$\mathrm{Isom}(X_1, X_2)$$

denotes the set of isomorphisms $X_1 \xrightarrow{\sim} X_2$ *of schemes;*

$$\mathrm{OutIsom}(\Pi_{X_1}, \Pi_{X_2})$$

denotes the set of outer isomorphisms $\Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}$ *of profinite groups [cf. Notations and conventions, Profinite groups].*

Note that, in light of Theorem 0.3, J. Stix's argument in [28] also gives a proof of Theorem A. On the other hand, our viewpoint gives an alternative proof of the J. Stix's result. One technical difficulty arises from the existence of Frobenius twists. [Note that the Frobenius twists may change the moduli of hyperbolic curves but may not change the étale fundamental groups of them.] In this situation, our key observation/philosophy is

> *such an obstruction that arises from the existence of Frobenius twists may be regarded as a(n) defect/indeterminacy of the compatibility/rigidity of the group-theoretic cyclotomes.*

Therefore, we do not control the Frobenius twists directly [as J. Stix did] but concentrate on the determination of a(n) defect/indeterminacy of compatibility/rigidity of the group-theoretic cyclotomes. One of the key ingredients of this determination is the following elementary observation [cf. [9], Theorem 1.2]:

> Let F be an algebraically closed field of characteristic p. Then the image of the graph
>
> $$\mathbb{A}^1(F) \longrightarrow \mathbb{A}^1(F) \times \mathbb{A}^1(F)$$
>
> associated to an automorphism $\alpha \in \mathrm{Aut}(F)$ is not Zariski-dense if and only if α is an integral power of the p-th Frobenius automorphism.

Moreover, in light of our viewpoint, by combining Theorem A with a certain enhanced version of a F. Pop's theorem in birational anabelian geometry, one may give a proof of Theorem B, i.e., an absolute version of the geometrically pro-Σ Grothendieck Conjecture for nonisotrivial hyperbolic curves over the perfections of finitely generated transcendental extension fields of finite fields. This absolute version may be regarded as a higher dimensional base field analogue of Theorem 0.3.

Before proceeding, let us observe that, in light of the theory of specialization, the compatibility/rigidity of the group-theoretic cyclotomes associated to the étale fundamental groups of hyperbolic curves over finitely generated fields of characteristic 0 is a direct consequence of the geometrically pro-prime-to-p version of the Grothendieck Conjecture for hyperbolic curves over finite fields of characteristic p, together with Chevalley's theorem [i.e., the discussion in the proof of [30], Claim (6.8), together with a variant of Lemma 2.7]. This observation leads to a proof of the Grothendieck Conjecture for hyperbolic curves over finitely generated fields of characteristic 0. On the other hand, in the characteristic 0 case, a much stronger result is obtained by S. Mochizuki as mentioned above. Therefore, in the remainder of the present paper, we do not discuss the Grothendieck Conjecture for hyperbolic curves in the case where the base fields are of characteristic 0.

Finally, in the present paper, we also investigate generalities concerning isotriviality [that is applied in the proof of Theorem A], especially, we establish the following isotriviality criteria for smooth curves [cf. Theorems 1.9, 1.14]:

Theorem C. *Let k be a field; $k \subseteq K$ a finitely generated field extension; X a smooth, proper curve over K of genus ≥ 1. Suppose that Σ does not contain the characteristic of k. Write*

$$\rho : G_K \longrightarrow \mathrm{Aut}(\Delta_X^{\mathrm{ab}})$$

for the natural Σ-adic Galois representation associated to X, where Δ_X^{ab} denotes the maximal abelian quotient of Δ_X. Then X is isotrivial relative to k if and only if $\rho(G_{K\cdot\overline{k}}) = \{1\}$.

Theorem D. *Let k be a field; $k \subseteq K$ a finitely generated field extension; X a hyperbolic curve over K. Suppose that Σ does not contain the characteristic of k. Write*

$$\rho : G_K \longrightarrow \mathrm{Out}(\Delta_X)$$

for the natural pro-Σ outer representation associated to X. Then X is isotrivial relative to k if and only if $\rho(G_{K\cdot\overline{k}}) = \{1\}$.

Here, we note that the assumption on the properness of X in Theorem C may not be dropped [cf. Remark 1.9.2]. These results may be of interest independent of anabelian geometry. Moreover, in light of the development of various criteria in arithmetic geometry such as good reduction criteria, it would be interesting to investigate the extent to which such isotriviality criteria exist for other classes of algebraic varieties or base fields. It may be worth mentioning that, by applying a similar argument to the argument applied in the proof of Theorem C, one may obtain a similar isotriviality criterion for abelian varieties over fields of characteristic 0 [cf. Remark 1.9.3]. Finally, as a corollary of Theorem D, one may obtain the following geometric result [cf. Corollary 1.17]:

Corollary E. *Let k be a field; $k \subseteq K$ a field extension; X a hyperbolic curve over K; S an algebraic variety over K that admits a dominant morphism $S \to X$ over K. Suppose that S is isotrivial relative to k. Then X is isotrivial relative to k.*

The present paper is organized as follows. In §1, by applying some basic properties of the K/k-trace associated to an abelian variety, Lang-Néron theorem, and a certain consequence of Martens' proof of Torelli's theorem, we first prove Theorem C. Next, by combining Theorem C with a certain trick concerning coverings of hyperbolic curves based on de Franchis-Severi theorem, we prove Theorem D. Moreover, by applying Theorem D, together with the slimness of surface groups, we prove Corollary E. In §2, from the viewpoint of the compatibility/rigidity of the group-theoretic cyclotomes, we formulate and prove a certain relative version of the Grothendieck Conjecture for hyperbolic curves over finitely generated extension fields of $\mathbb{F}_p$. Note that, in the positive characteristic case, the relative version of the Grothendieck Conjecture [in the usual sense] does not hold even if we assume that the base field is finite. This is the reason why we introduce a modified formulation in this section. On the other hand, the results in the present section may be regarded as byproducts of highly nontrivial results that have been obtained by Tamagawa, Mochizuki, and Saïdi. In §3, we prove that the kernel of the homomorphism from the automorphism group of the perfection of a positive characteristic discrete valuation field to the outer automorphism group of the absolute Galois group of the field consists of the Frobenius automorphisms. In §4, in light of the Saïdi-Tamagawa's result, together with the isotriviality criteria in §1; [9], Theorem 1.2, we first determine the defect/indeterminacy of compatibility/rigidity of the group-theoretic cyclotomes. As a direct consequence of this determination and the modified relative version of the Grothendieck Conjecture formulated and proved in §2, we give a proof of Theorem A, i.e., an alternative proof of the Stix's result. Finally, by combining Theorem A with a Pop's theorem in birational anabelian geometry, together with the result obtained in §3, we prove Theorem B.

Notations and conventions

Numbers: The notation $\mathfrak{Primes}$ will be used to denote the set of prime numbers. The notation $\mathbb{Z}$ will be used to denote the ring of integers. The notation $\widehat{\mathbb{Z}}$ will be used to denote the profinite completion of the underlying additive group of $\mathbb{Z}$. If p is a prime number, then the notation $\mathbb{Z}_p$ will be used to denote the maximal pro-p quotient of $\widehat{\mathbb{Z}}$; the notation $\widehat{\mathbb{Z}}^{(p)'}$ will be used to denote the maximal pro-prime-to-p quotient of $\widehat{\mathbb{Z}}$; the notation $\mathbb{F}_p$ will be used to denote the finite field of cardinality p.

Fields: Let F be a field. Then we shall write $\overline{F}$ for the algebraic closure [determined up to isomorphisms] of F; F^{sep} $(\subseteq \overline{F})$ for the separable closure of F; $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(F^{\mathrm{sep}}/F)$; F^{pf} for the perfection of F.

Profinite groups: Let G be a profinite group; Σ a nonempty set of prime number; l a prime number. Then we shall write G^{ab} for the abelianization of G, i.e., the quotient of G by the closure of the commutator subgroup $[G,G] \subseteq G$; G^{Σ} for the maximal pro-Σ quotient of G; $G^{l} \stackrel{\mathrm{def}}{=} G^{\{l\}}$; $\mathrm{Aut}(G)$ for the group of continuous automorphisms of the profinite group G; $\mathrm{Inn}(G)$ $(\subseteq \mathrm{Aut}(G))$ for the group of inner automorphisms of G; $\mathrm{Out}(G) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(G)/\mathrm{Inn}(G)$.

Let G_1, G_2 be profinite groups; $H_2 \subseteq G_2$ a closed subgroup; $\phi : G_1 \xrightarrow{\sim} G_2$ a continuous isomorphism of profinite groups considered up to composition with the inner automorphism determined by an element $\in H_2$. Then we shall refer to such ϕ as an H_2-*outer isomorphism.* In the case where $H_2 = G_2$, we shall refer to such ϕ as an *outer isomorphism.* [Note that, if $G_1 = G_2 = H_2$, then ϕ may be regarded as an element of $\mathrm{Out}(G_1)$.]

Schemes: Let F be a field; X an algebraic variety [i.e., separated, geometrically integral, and of finite type scheme] over F; $F \subseteq E$ a field extension. Then we shall write $X_E \stackrel{\mathrm{def}}{=} X \times_F E$; $X(E)$ for the set of E-valued points of X.

Fundamental groups: Let S be a connected Noetherian scheme. Then we shall write $\pi_1^{\text{ét}}(S)$ for the étale fundamental group of S, relative to a suitable choice of basepoint [cf. [7]]. Suppose that S is an open subscheme of a connected regular scheme $\overline{S}$ whose complement $D_S \stackrel{\mathrm{def}}{=} \overline{S} \setminus S$ is a normal crossing divisor on $\overline{S}$. Then we shall write $\pi_1^{\mathrm{tame}}(S) \stackrel{\mathrm{def}}{=} \pi_1^{\mathrm{tame}}(\overline{S}, D_S)$ for the tame fundamental group of S [that classifies the finite étale coverings of S tamely ramified along D_S], relative to a suitable choice of basepoint [cf. [6], Corollary 2.4.4]. In particular, $\pi_1^{\mathrm{tame}}(S)$ may be regarded as a quotient of $\pi_1^{\text{ét}}(S)$. Here, we note that $\pi_1^{\mathrm{tame}}(S)$ depends on the choice of $\overline{S}$. On the other hand, throughout the present paper, there is no confusion on the choice of such $\overline{S}$. Therefore, we do not specify $\overline{S}$ in the notation of $\pi_1^{\mathrm{tame}}(S)$.

1 Isotriviality criteria for hyperbolic curves

In the present section, from the viewpoint of both Galois and outer Galois representation, we give certain isotriviality criteria for hyperbolic curves. The key ingredients of the proof are some basic properties of the K/k-trace associated to an abelian variety, Lang-Néron theorem, and a certain consequence of Martens' proof of Torelli's theorem.

First, we begin by recalling the definitions of hyperbolic curves over schemes and isotriviality of algebraic varieties.

Definition 1.1. Let S be a connected, normal scheme; X a smooth curve over S [i.e., smooth, separated, and of finite type scheme over S whose geometric fibers are integral and of dimension 1]. Then we shall say that X is a *hyperbolic curve* over S if X admits a smooth compactification $(X \subseteq)$ $\overline{X}$ over S satisfying the following conditions:

- The complement $\overline{X} \setminus X \subseteq \overline{X}$ of X is a relative effective Cartier divisor on $\overline{X}/S$ that is étale over S.
- If we write g for the genus of the smooth, proper curve $\overline{X}$ over S; r for the degree of the finite étale morphism $\overline{X} \setminus X \subseteq X \to S$, then it holds that $2g - 2 + r > 0$.

Remark 1.1.1. It follows immediately from the discussion in [18], §0, Curves, that the smooth compactification $\overline{X}$ of X over S satisfying the above two conditions is unique.

Definition 1.2. Let F be a field; $F_0 \subseteq F$ a subfield; X an algebraic variety over F. Then we shall say that X is *isotrivial* relative to F_0 if $X_{\overline{F}}$ descends to an algebraic variety over $\overline{F}_0$.

Next, we recall some basic properties of the K/k-traces associated to abelian varieties, together with the well-known Lang-Néron theorem.

Definition 1.3. Let k be a field; $k \subseteq K$ a field extension. Then we shall say that:

(i) The field extension $k \subseteq K$ is *primary* if the algebraic closure of k in K is purely inseparable over k.

(ii) The field extension $k \subseteq K$ is *regular* if $k \subseteq K$ is separable, and k is algebraically closed in K.

Remark 1.3.1. Note that it follows immediately from the definitions that every regular field extension is primary. Note also that every field extension of an algebraically closed field is regular [cf. [4], Corollary 2.6.5, (c)].

Definition 1.4. Let k be a field; $k \subseteq K$ a primary field extension; A an abelian variety over K. Then a K/k-trace $(\mathrm{Tr}_{K/k}(A), \tau)$ associated to A is a final object in the category of pairs of an abelian variety B over k and a morphism $B_K \to A$ of abelian varieties over K [cf. [1], Definition 6.1].

Theorem 1.5. *Let k be a field; $k \subseteq K$ a primary field extension; A an abelian variety over K. Then the K/k-trace associated to A exists [cf. [1], Theorem 6.2]. Write*

$$\tau : \mathrm{Tr}_{K/k}(A)_K \longrightarrow A$$

for the K/k-trace associated to A. Then the following hold:

(i) There exists a unique abelian subvariety $A' \subseteq A$ such that $\mathrm{Tr}_{K/k}(A/A')$ is the trivial abelian variety, and the K/k-trace $\mathrm{Tr}_{K/k}(A')_K \to A'$ associated to A' is an isogeny [cf. [1], Theorem 6.4].

(ii) Suppose that the field extension $k \subseteq K$ is regular. Then $\mathrm{Ker}(\tau)$ is a connected finite group scheme over K with a connected Cartier dual [cf. [1], Theorem 6.12].

(iii) Suppose that the field extension $k \subseteq K$ is finitely generated and regular. Then $A(K)/\mathrm{Tr}_{K/k}(A)(k)$ is finitely generated [cf. [1], Theorem 7.1; [11], Theorem 1].

As an application of Theorem 1.5, together with a variant of Torelli's theorem [cf. Proposition 1.8 below], we give a certain monodromy criterion of isotriviality for smooth, proper curves of genus ≥ 1 [cf. Theorem 1.9 below].

Lemma 1.6. *Let k be an algebraically closed field; $k \subseteq K$ a finitely generated field extension of transcendental degree 1; X a proper hyperbolic curve over K; S a smooth curve over k whose function field coincides with K. Suppose that*

- *the proper hyperbolic curve X over K extends to a proper hyperbolic curve $\mathcal{X}$ over S, and*
- *there exists an infinite subset $I \subseteq S(k)$ such that the special fibers of $\mathcal{X}$ at the closed points $\in I$ are isomorphic over k.*

Then X is isotrivial relative to k.

Proof. Fix a closed point $s_0 \in I$, and write $\mathcal{X}_{s_0}$ for the special fiber of $\mathcal{X}$ at s_0. In particular, for each $s \in I$, the special fibers of the proper hyperbolic curves $\mathcal{X}_{s_0} \times_k S$, $\mathcal{X}$ at s are isomorphic. Write $\mathbf{Isom}_S(\mathcal{X}_{s_0} \times_k S, \mathcal{X})$ for the Isom scheme determined by the proper hyperbolic curves $\mathcal{X}_{s_0} \times_k S$, $\mathcal{X}$ over S; ϕ for the structure morphism of this Isom scheme. Recall that ϕ is finite [cf. [2], Theorem 1.11], hence closed. On the other hand, it follows immediately from the definition of I that the image of ϕ contains the subset $I \subseteq S(k)$. Here, we note that since S is a smooth curve over k, the infinite subset I of closed points is dense in S. In particular, the closedness of ϕ implies that ϕ is surjective. Thus, since ϕ is finite, we conclude that there exists a finite field extension $K \subseteq L$ such that $(\mathcal{X}_{s_0})_L$ is isomorphic to X_L over L. This completes the proof of Lemma 1.6. □

In the remainder of the present section, for each abelian scheme $\mathcal{A}$ over a scheme, we shall write $\mathcal{A}^\vee$ for the dual abelian scheme of $\mathcal{A}$. For each smooth, proper curve $\mathcal{X}$ over a scheme, we shall write $J(\mathcal{X})$ for the relative Jacobian of $\mathcal{X}$.

Lemma 1.7. *Let k be an algebraically closed field; A an abelian variety over k equipped with a purely inseparable polarization; S a smooth curve over k; $\mathcal{B}$ a principally polarized abelian scheme over S;*

$$f : A \times_k S \longrightarrow \mathcal{B}$$

a [purely inseparable] isogeny of polarized abelian schemes over S. For $i = 1, 2$, let $s_i \in S(k)$ be a closed point and write $B_i \stackrel{\mathrm{def}}{=} \mathcal{B} \times_S s_i$;

$$f_i : A \longrightarrow B_i$$

for the [purely inseparable] isogeny of polarized abelian varieties over k induced by f. Suppose that the principal polarization $\mathcal{B} \xrightarrow{\sim} \mathcal{B}^\vee$ on $\mathcal{B}$ is determined by a(n) [integral] relative effective Cartier divisor $\mathcal{D} \subseteq \mathcal{B}$ on $\mathcal{B}/S$ with geometrically integral fibers. For $i = 1, 2$, write $D_i \stackrel{\mathrm{def}}{=} \mathcal{D} \times_S s_i$. Write

$$D' \stackrel{\mathrm{def}}{=} f_2(f_1^{-1}(D_1))^{\mathrm{red}} \subseteq B_2.$$

Then D' is a translate of D_2 via a closed point of B_2.

Proof. First, since f is an isogeny of polarized abelian schemes, the polarization $A \times_k S \to A^\vee \times_k S$ on $A \times_k S$ determined by the irreducible relative effective Cartier divisor $f^{-1}(\mathcal{D}) \subseteq A \times_k S$ on $A \times_k S/S$ coincides with the polarization induced by the given polarization on A. Write K for the function field of S; $\mathcal{C}_1 \stackrel{\mathrm{def}}{=} f_1^{-1}(D_1) \times_k S$; $\mathcal{L}(-)$ for the line bundle associated to an effective Cartier divisor $(-)$. Recall that the Neron-Severi group associated to $A_{\overline{K}}$ is embedded into $\mathrm{Hom}(A_{\overline{K}}, A^\vee_{\overline{K}})$. Then since $\mathcal{L}(\mathcal{C}_1)$ and $\mathcal{L}(f^{-1}(\mathcal{D}))$ determine a same polarization on $A \times_k S$, after possibly replacing K by a finite extension field of K, we observe that there exists an S-valued point $x \in A \times_k S$ such that

$$(\mathcal{L}(T_x^{-1}\mathcal{C}_1) =)\ T_x^*\mathcal{L}(\mathcal{C}_1) \cong \mathcal{L}(f^{-1}(\mathcal{D})),$$

where $T_x : A \times_k S \xrightarrow{\sim} A \times_k S$ denotes the translate automorphism determined by x. Write $m \stackrel{\mathrm{def}}{=} \deg f$; μ_1 (respectively, $\mu_{\mathcal{C}_1}$) for the multiplicity of the irreducible effective Cartier divisor $f_1^{-1}(D_1) \subseteq A$ (respectively, $\mathcal{C}_1 \times_S K = f_1^{-1}(D_1) \times_k K \subseteq A_K$);

$$\phi : \mathcal{C}_1^{\mathrm{red}} \longrightarrow f(\mathcal{C}_1)^{\mathrm{red}}, \quad \phi_{s_1} : f_1^{-1}(D_1)^{\mathrm{red}} \longrightarrow D_1$$

for the natural finite morphisms. Here, we note that:

- $\mu_{\mathcal{C}_1} = \mu_1$, and $m = \deg f_2 = \deg f_1 = \mu_1 \cdot \deg \phi_{s_1}$.

- The residue field extension of ϕ at the points corresponding to the generic points of $\mathcal{C}_1^{\mathrm{red}} \times_S s_1$, $f(\mathcal{C}_1)^{\mathrm{red}} \times_S s_1$ coincides with the function field extension of ϕ_{s_1}.

In particular, $m_1 \stackrel{\mathrm{def}}{=} \mu_{\mathcal{C}_1} \cdot \deg \phi \geq m$. On the other hand, we recall that the push-forward by a finite morphism between algebraic varieties preserves linear equivalence relations on Weil divisors. Therefore, since $\mathcal{L}(T_x^{-1}\mathcal{C}_1) \cong \mathcal{L}(f^{-1}(\mathcal{D}))$, by considering the push-forward by f, we observe that

$$\mathcal{L}(f(T_x^{-1}\mathcal{C}_1)^{\mathrm{red}})^{\otimes m_1} \cong \mathcal{L}(\mathcal{D})^{\otimes m}.$$

Then there exists a positive integer $m_2 \geq m_1$ such that

$$\mathcal{L}(f_2(T_x^{-1}\mathcal{C}_1 \times_S s_2)^{\mathrm{red}})^{\otimes m_2} \cong \mathcal{L}(D_2)^{\otimes m}.$$

Thus, since $m_2 \geq m$, and D_2 determines a principal polarization, by considering the respective degrees of the isogenies associated to the ample line bundles $\mathcal{L}(f_2(T_x^{-1}\mathcal{C}_1 \times_S s_2)^{\mathrm{red}})^{\otimes m_2}$, $\mathcal{L}(D_2)^{\otimes m}$ on B_2, we conclude that $m = m_2$. Moreover, in light of the torsion-freeness of the Neron-Severi group associated to B_2, this implies that

$$\mathcal{L}(f_2(T_x^{-1}\mathcal{C}_1 \times_S s_2)^{\mathrm{red}}) \cong \mathcal{L}(D_2)$$

Note that since $D_2 \subseteq B_2$ is an effective Cartier divisor, $H^0(B_2, \mathcal{L}(D_2)) \neq \{0\}$. Then since D_2 determines a principal polarization, it follows immediately from theorems in [23], p. 150, that $H^0(B_2, \mathcal{L}(D_2)) = k$. Thus, we conclude that $f_2(T_x^{-1}\mathcal{C}_1 \times_S s_2)^{\mathrm{red}} = D_2$, hence that D' is a translate of D_2. This completes the proof of Lemma 1.7. □

Now we prove a certain consequence of Martens' proof of Torelli's theorem.

Proposition 1.8. *Let k be an algebraically closed field; A an abelian variety over k equipped with a purely inseparable polarization*

$$\phi : A \longrightarrow A^{\vee};$$

S a smooth curve over k; $\mathcal{X}$ a proper hyperbolic curve or an elliptic curve over S. Suppose that there exists a(n) [purely inseparable] isogeny

$$f : A \times_k S \longrightarrow J(\mathcal{X})$$

of polarized abelian schemes over S, where we regard $J(\mathcal{X})$ as a polarized abelian scheme via the canonical polarization [cf. [3], Proposition 6.9]. Then the generic fiber of $\mathcal{X}$ is isotrivial relative to k.

Proof. First, suppose that $\mathcal{X}$ is an elliptic curve. Then $\mathcal{X} \xrightarrow{\sim} J(\mathcal{X})$, and A is also an elliptic curve. Note that, for each point s of S, the dual isogeny of f induces a purely inseparable isogeny $J(\mathcal{X}) \times_S s \to A^{\vee}$. In particular, since $A^{\vee}$ is a smooth, proper curve, the generic fiber of $J(\mathcal{X})$ descends [up to isomorphisms] to a Frobenius twist of $A^{\vee}$. Thus, we conclude that the generic fiber of $\mathcal{X}$ is isotrivial relative to k.

Next, suppose that $\mathcal{X}$ is a proper hyperbolic curve. In light of Lemma 1.6, it suffices to verify that, up to isomorphisms, only finitely many proper hyperbolic curves appear in the special fibers of $\mathcal{X}$. Let s_1, s_2 be closed points of S. For $i = 1, 2$, write $X_i \stackrel{\mathrm{def}}{=} \mathcal{X} \times_S s_i$;

$$\iota_i : X_i \hookrightarrow J(X_i)$$

for the Albanese embedding relative to a point $\in X_i(k)$; $g \geq 2$ for the genus of X_i;

$$\theta_i$$

for the effective Cartier divisor on $J(X_i)$ that appears as the scheme-theoretic image of the natural morphism $X_i^{(g-1)} \to J(X_i)$ induced by ι_i. Then, for $i = 1, 2$, f induces a purely inseparable isogeny

$$\psi_i : A \longrightarrow J(X_i)$$

such that $\psi_i^{-1}(\theta_i)$ is an irreducible effective Cartier divisor that determines the polarization ϕ. Note that since the isogenies ψ_1 and ψ_2 are purely inseparable, we have the composite of isomorphisms

$$\psi : J(X_2)(k) \overset{\sim}{\leftarrow} A(k) \overset{\sim}{\rightarrow} J(X_1)(k)$$

of the groups of k-valued points. Recall that the canonical polarizations on Jacobians are principal. Therefore, it follows immediately from Lemma 1.7, together with the various definitions involved, that $\psi(\theta_2(k))$ is a translate of $\theta_1(k)$ via a closed point of $J(X_1)$. Moreover, in light of the construction of ψ, we observe that ψ maps any Zariski closed subset to a Zariski closed subset of same dimension. Thus, we conclude from Martens' proof of Torelli's theorem [cf. [12]; [13], §13] that $\psi(\iota_2(X_2)(k))$ is a translate of $\iota_1(X_1)(k)$ or of $\iota_1(X_1)^*(k)$, where $\iota_1(X_1)^*$ denotes the reflection of $\iota_1(X_1)$ in $J(X_1)$. Next, for $i = 1, 2$, since ϕ is a purely inseparable polarization, and ψ_i is a purely inseparable isogeny of polarized abelian varieties, we obtain a purely inseparable isogeny

$$\psi_i^\vee : J(X_i) \longrightarrow A^\vee.$$

such that $\psi_i^\vee \circ \psi_i = \phi$. Here, we note that $\psi_1^\vee$ and $\psi_2^\vee$ are compatible with ψ in a natural sense. For $i = 1, 2$, write

$$Z_i \subseteq A^\vee$$

for the scheme-theoretic image of the composite morphism $\psi_i^\vee \circ \iota_i : X_i \to A^\vee$. Then since $\psi(\iota_2(X_2)(k))$ is a translate of $\iota_1(X_1)(k)$ or $\iota_1(X_1)^*(k)$, it holds that Z_2 is a translate of Z_1 or the reflection of Z_1. In particular, Z_1 and Z_2 are isomorphic over k. On the other hand, observe that the degrees of the finite purely inseparable extensions on function fields associated to the finite morphisms $X_1 \to Z_1$, $X_2 \to Z_2$ are bounded by deg ϕ.

In summary, there exists a one-variable function field K over k such that the function field of any proper hyperbolic curve over k that arises as a special fiber of $\mathcal{X}$ is isomorphic to a finite purely inseparable extension field of K of degree $\leq$ deg ϕ over k. Thus, since the isomorphism class of a finite purely inseparable extension field of a one-variable function field over k is determined by its degree, we conclude that, up to isomorphisms, only finitely many proper hyperbolic curves appear in the special fibers of $\mathcal{X}$. This completes the proof of Proposition 1.8. $\square$

Theorem 1.9. *Let k be a field; $k \subseteq K$ a finitely generated field extension; X a smooth, proper curve over K of genus ≥ 1; Σ a nonempty set of prime numbers invertible in k. Write $\Delta_X \overset{\mathrm{def}}{=} \pi_1^{\text{ét}}(X_{\overline{K}})^\Sigma$;*

$$\rho : G_K \longrightarrow \mathrm{Aut}(\Delta_X^{\mathrm{ab}})$$

for the natural Σ-adic Galois representation associated to X. Then X is isotrivial relative to k if and only if $\rho(G_{K \cdot \overline{k}}) = \{1\}$.

Proof. Necessity is immediate. To verify sufficiency, it suffices to consider the case where $\Sigma = \{l\}$ for some prime number l. Suppose that $\rho(G_{K \cdot \overline{k}}) = \{1\}$. Then, by replacing k, K by $\overline{k}$, a suitable finite extension field of $K \cdot \overline{k}$, respectively, we may assume without loss of generality that

$$k = \overline{k}, \quad X(K) \neq \emptyset, \quad \rho(G_K) = \{1\}.$$

In particular, the field extension $k \subseteq K$ is regular [cf. Remark 1.3.1]. Moreover, in light of the induction on the transcendental degree of the field extension $k \subseteq K$, we may assume without loss of generality that K is a one-variable function field over k. Write $J(X)' \subseteq J(X)$ for the unique subabelian variety as in Theorem 1.5, (i). Then since $\rho(G_K) = \{1\}$, it follows immediately from Theorem 1.5, (i), (iii), together with [21], Proposition A.6, (iv), that $J(X)' = J(X)$, and the K/k-trace

$$\tau : \mathrm{Tr}_{K/k}(J(X))_K \longrightarrow J(X)$$

associated to $J(X)$ is an isogeny over K. Furthermore, it follows from Theorem 1.5, (ii), that τ is a purely inseparable isogeny over K whose dual is also purely inseparable. Next, let S be a smooth curve over k such that

- the function field of S coincides with K,
- X extends to a smooth, proper curve $\mathcal{X}$ over S, and
- τ extends to an isogeny

$$\tilde{\tau} : \mathrm{Tr}_{K/k}(J(X))_S \longrightarrow J(\mathcal{X})$$

  of abelian schemes over S.

Then since τ is a purely inseparable isogeny over K whose dual is also purely inseparable, we observe that $\tilde{\tau}$, the dual of $\tilde{\tau}$, and the canonical polarization of $J(\mathcal{X})$ determine a purely inseparable polarization

$$\tilde{\phi} : \mathrm{Tr}_{K/k}(J(X))_S \longrightarrow \mathrm{Tr}_{K/k}(J(X))_S^{\vee}$$

over S. Note that, in light of Chow's theorem [cf. [1], Theorem 3.19], $\tilde{\phi}$ arises from a purely inseparable polarization

$$\phi : \mathrm{Tr}_{K/k}(J(X)) \longrightarrow \mathrm{Tr}_{K/k}(J(X))^{\vee}$$

over k. Thus, we conclude from Proposition 1.8 that X is isotrivial relative to k. This completes the proof of sufficiency, hence of Theorem 1.9. $\square$

Remark 1.9.1. In the notation of Theorem 1.9, suppose that X is nonisotrivial relative to k, and $\Sigma = \{l\}$ for some prime number l invertible in k. Let $H \subseteq G_{K \cdot \overline{k}}$ be an open subgroup. Then it follows immediately from Theorem 1.9, together with the various definitions involved, that $\rho(H) \neq \{1\}$. On the other hand, we recall that the kernel of the natural homomorphism $\mathrm{Aut}(\Delta_X^{\mathrm{ab}}) \to \mathrm{Aut}(\Delta_X^{\mathrm{ab}}/l^2 \cdot \Delta_X^{\mathrm{ab}})$ is a torsion-free pro-l group [cf. Lemma 1.10 below]. In particular, after replacing K by a finite extension field of K, one may observe that $\rho(G_{K \cdot \overline{k}})$ is a nontrivial torsion-free pro-l group.

Remark 1.9.2. The assumption that X is proper over K in Theorem 1.9 may not be dropped. Indeed, suppose that k is algebraically closed, and the field extension $k \subseteq K$ is transcendental. Let X_0 be a proper hyperbolic curve over k; $x \in X_0(K) \setminus X_0(k)$. Observe that the abelianizations of the respective geometric pro-Σ fundamental groups of a once-punctured hyperbolic curve and its smooth compactification are isomorphic naturally. Then, in light of this observation, the nonisotrivial affine hyperbolic curve $X_0 \times_k K \setminus \{x\}$ over K gives a counter-example.

Remark 1.9.3. Let k be a field of characteristic 0; $k \subseteq K$ a finitely generated field extension; A an abelian variety over K; Σ a nonempty set of prime numbers. Write $T_\Sigma A$ for the Σ-adic Tate module associated to A [i.e., the product of l-adic Tate modules associated to A, where l ranges over the prime numbers $\in \Sigma$];

$$\rho : G_K \longrightarrow \mathrm{Aut}(T_\Sigma A)$$

for the natural Σ-adic Galois representation associated to A. Then it follows from a similar argument to the argument applied in the proof of Theorem 1.9 that A is isotrivial relative to k if and only if $\rho(G_{K \cdot \bar{k}}) = \{1\}$.

Lemma 1.10. *Let l be a prime number; n a positive integer. Suppose that l is odd (respectively, $l = 2$). Then the kernel of the natural surjection*

$$\phi : GL_n(\mathbb{Z}_l) \twoheadrightarrow GL_n(\mathbb{F}_l) \quad (\text{respectively, } \phi : GL_n(\mathbb{Z}_l) \twoheadrightarrow GL_n(\mathbb{Z}/l^2\mathbb{Z}))$$

is torsion-free.

Proof. Since $\mathrm{Ker}(\phi)$ is a pro-l group, it suffices to verify that $\mathrm{Ker}(\phi)$ has no nontrivial l-torsion element. Write $I_n \in GL_n(\mathbb{Z}_l)$ for the identity matrix. Let $A \in \mathrm{Ker}(\phi)$ be such that $A^l = I_n$. Suppose that $A \neq I_n$. Then there exist a positive integer t (respectively, $t \geq 2$) and a matrix $B \in M_n(\mathbb{Z}_l) \setminus M_n(l\mathbb{Z}_l)$ such that $A = I_n + l^t B$. Therefore, since $A^l = I_n$, it holds that

$$\sum_{1 \leq i \leq l} \binom{l}{i} \cdot l^{it} B^i = 0.$$

Write $a_i \stackrel{\text{def}}{=} \binom{l}{i} \cdot l^{it}$; v_l for the l-adic additive valuation such that $v_l(l) = 1$. Then it follows immediately from our assumption that l is odd (respectively, $l = 2$, and $t \geq 2$) that $v_l(a_l) = lt > t + 1 = v_l(a_1)$, and $v_l(a_i) = it + 1 > t + 1 = v_l(a_1)$ for each $i = 2, \ldots, l - 1$. In particular, it follows from the equality in the above display that $B \in M_n(l\mathbb{Z}_l)$. This contradicts the condition that $B \in M_n(\mathbb{Z}_l) \setminus M_n(l\mathbb{Z}_l)$. Thus, we conclude that $A = I_n$. This completes the proof of Lemma 1.10. $\square$

In light of Remark 1.9.2, to establish an isotriviality criterion for affine hyperbolic curves, we consider an anabelian setting. To obtain an anabelian criterion of isotriviality for hyperbolic curves, we prepare some lemmas.

Lemma 1.11. *Let k be a separably closed field; $k \subseteq K$ a field extension; X, Y proper hyperbolic curves over k. Then any isomorphism $X_K \xrightarrow{\sim} Y_K$ over K descends to an isomorphism $X \xrightarrow{\sim} Y$ over k. In particular, if a proper hyperbolic curve Z over K is nonisotrivial relative to k, then for each field extension $K \subseteq L$, Z_L is also nonisotrivial relative to k.*

Proof. Since k is separably closed, Lemma 1.11 follows immediately from the fact that the Isom scheme $\mathbf{Isom}_k(X, Y)$ is finite and unramified over k [cf. [2], Theorem 1.11]. $\square$

Remark 1.11.1. Let k be a field; $k \subseteq K$ a field extension; X a hyperbolic curve over K; $Z \to X$ a [possibly, ramified] finite Galois covering over K whose degree is coprime to the characteristic of k. Suppose that Z is isotrivial relative to k and of genus ≥ 2. In particular, the smooth compactification of Z [obtained as the normalization of the smooth compactification of X in the function field of Z] is a proper hyperbolic curve that is isotrivial relative to k. Then it follows immediately from Lemma 1.11 that X is also isotrivial relative to k. A much more general statement will be given later [cf. Corollary 1.17].

Now we recall the well-known de Franchis-Severi theorem.

Theorem 1.12. *Let k be an algebraically closed field; X, Y proper hyperbolic curves over k. Then the cardinality of the set of generically étale, finite morphisms $Y \to X$ over k is finite.*

Proof. Theorem 1.12 follows immediately from [25], p.49, Theorem. □

Lemma 1.13. *Let k be a field; l a prime number invertible in k; $k \subseteq K$ a finitely generated field extension; X an affine hyperbolic curve over K. Suppose that X is nonisotrivial relative to k. Then there exists a geometrically pro-l finite étale Galois covering $Y \to X$ such that the smooth compactification of Y is nonisotrivial relative to k.*

Proof. First, in light of Hurwitz's formula, together with Remark 1.11.1, by replacing X by the domain curve of a suitable geometrically pro-l finite étale Galois covering of X, we may assume without loss of generality that X has genus ≥ 2. Next, in light of the induction on the transcendental degree of the finitely generated field extension $k \subseteq K$, together with Lemma 1.11 and the various definitions involved, that we may assume without loss of generality that:

- k is algebraically closed, and the transcendental degree of the field extension $k \subseteq K$ is 1.
- The smooth compactification $\overline{X}$ of X over K is isotrivial relative to k.
- If we write $\overline{X}_0$ for the proper hyperbolic curve over k such that $(\overline{X}_0)_K$ is isomorphic to $\overline{X}$ over K, then the divisor $\overline{X} \setminus X \subseteq \overline{X}$ consists of a K-valued point that does not arise from a k-valued point of $\overline{X}_0$.

After replacing K by a finite separable extension field of K, if necessary, we fix a finite étale Galois covering $Y \to X$ over K of degree l that ramifies over the K-valued point in the complement of X. Then it suffices to verify that the smooth compactification $\overline{Y}$ of Y over K is nonisotrivial relative to k [cf. also Remark 1.11.1].

Suppose that $\overline{Y}$ is isotrivial relative to k. Write $\overline{Y}_0$ for the proper hyperbolic curve over k such that $(\overline{Y}_0)_K$ is isomorphic to $\overline{Y}$ over K. Let S_0 be a smooth, proper curve over k whose function field coincides with K. Then the respective K-valued points in the complements of X, Y determine respective sections

$$\iota_X : S_0 \longrightarrow \overline{X}_0 \times_k S_0, \quad \iota_Y : S_0 \longrightarrow \overline{Y}_0 \times_k S_0$$

of the second projection morphisms, and the finite étale Galois covering $Y \to X$ over K extends to a finite étale Galois covering

$$\phi : \overline{Y}_0 \times_k S_0 \setminus \mathrm{Im}(\iota_Y) \longrightarrow \overline{X}_0 \times_k S_0 \setminus \mathrm{Im}(\iota_X)$$

over S_0. For each closed point $s \in S_0$, write

$$\overline{\phi}_s : \overline{Y}_0 \longrightarrow \overline{X}_0$$

for the finite morphism over k determined by the finite étale Galois covering between the special fibers at s induced by ϕ. Then it follows immediately from Theorem 1.12 that there exists an infinite subset $I \subseteq S_0(k)$ such that $\overline{\phi}_{s_1} = \overline{\phi}_{s_2}$ for any $s_1, s_2 \in I$. Write $\overline{\phi}_I \stackrel{\mathrm{def}}{=} \overline{\phi}_s$, where $s \in I$. On the other hand, since the K-valued point in the complement of X does not arise from a k-valued point of $\overline{X}_0$, we observe that the composite of ι_X with the first projection morphism $\overline{X}_0 \times_k S_0 \to \overline{X}_0$ is a dominant morphism. Thus, since ϕ ramifies at the divisor $\mathrm{Im}(\iota_X)$, we conclude that the finite morphism $\overline{\phi}_I$ ramifies at infinitely many closed points. This is a contradiction. Thus, we conclude that $\overline{Y}$ is nonisotrivial relative to k. This completes the proof of Lemma 1.13. □

Next, in light of Lemma 1.13, we give an anabelian criterion of isotriviality for hyperbolic curves from the viewpoint of outer representations.

Theorem 1.14. *Let k be a field; $k \subseteq K$ a finitely generated field extension; X a hyperbolic curve over K; Σ a nonempty set of prime numbers invertible in k. Write $\Delta_X \stackrel{\mathrm{def}}{=} \pi_1^{\text{ét}}(X_{\overline{K}})^{\Sigma}$;*

$$\rho : G_K \longrightarrow \mathrm{Out}(\Delta_X)$$

for the natural pro-Σ outer representation associated to X. Then X is isotrivial relative to k if and only if $\rho(G_{K\cdot\overline{k}}) = \{1\}$.

Proof. Necessity is immediate. To verify sufficiency, it suffices to consider the case where $\Sigma = \{l\}$ for some prime number l. Suppose that X is nonisotrivial relative to k. Then it follows from Lemma 1.13 that there exists a geometrically pro-l finite étale Galois covering $Y \to X$ such that the smooth compactification of the hyperbolic curve Y [over a finite extension field of K] is nonisotrivial relative to k. After replacing K by a finite extension field of K, write

$$\rho_Y : G_K \longrightarrow \mathrm{Out}(\Delta_Y)$$

for the natural pro-l outer representation associated to Y, where $\Delta_Y \stackrel{\mathrm{def}}{=} \pi_1^{\text{ét}}(Y_{\overline{K}})^l$. Thus, we conclude from Theorem 1.9, together with Remark 1.9.1, that $\rho_Y(G_{K\cdot\overline{k}})$ is not *finite*. Therefore, since the index of the normal closed subgroup $\Delta_Y \subseteq \Delta_X$ is *finite*, it holds that $\rho(G_{K\cdot\overline{k}}) \neq \{1\}$. This completes the proof of sufficiency, hence of Theorem 1.14. $\square$

Here, we recall the definition and an elementary property of slimness of profinite groups, which play important roles in anabelian geometry.

Definition 1.15. Let G be a profinite group. Then we shall say that G is *slim* if every open subgroup of G is center-free.

Lemma 1.16. *Let G be a slim profinite group; $H \subseteq G$ an open subgroup; $\sigma \in \mathrm{Aut}(G)$. Suppose that σ induces the identity automorphism of H. Then σ is the identity automorphism.*

Proof. By replacing H by a smaller open subgroup of G, we may assume without loss of generality that H is normal in G. Then since G is slim, the natural homomorphism $f : G \to \mathrm{Aut}(H)$ obtained by taking conjugates is injective. On the other hand, we observe that the respective inner automorphisms of G, $\mathrm{Aut}(H)$ determined by σ, $\sigma|_H$ are compatible with respect to the natural homomorphism f. Thus, since f is injective, we conclude from our assumption that $\sigma|_H$ is the identity automorphism that σ is also the identity automorphism. This completes the proof of Lemma 1.16. $\square$

Finally, as an application of Theorem 1.14, we prove the following geometric result.

Corollary 1.17. *Let k be a field; $k \subseteq K$ a field extension; X a hyperbolic curve over K; S an algebraic variety over K that admits a dominant morphism $S \to X$ over K. Suppose that S is isotrivial relative to k. Then X is isotrivial relative to k.*

Proof. First, by replacing K by a suitable subfield of K, we may assume without loss of generality that the field extension $k \subseteq K$ is finitely generated. Next, let l be a prime number invertible in k. Write $\Delta_{(-)} \stackrel{\mathrm{def}}{=} \pi_1^{\text{ét}}((-)_{\overline{K}})^l$. Recall that Δ_X is slim [cf. [22], Proposition 1.4]. Then, in light of Theorem 1.14, together with Lemma 1.16, it suffices to verify that the natural homomorphism $f : \Delta_S \to \Delta_X$ induced by the dominant morphism $S \to X$ is open. On the other hand, if $M \subseteq L$ is a finitely generated field extension, then the natural homomorphism $G_L \to G_M$ is an open homomorphism. Thus, since the dominant morphism $S \to X$ induces a finitely generated field extension of function fields, we conclude that f is open. This completes the proof of Corollary 1.17. $\square$

2 Modified formulation of the relative Grothendieck Conjecture for hyperbolic curves over fields of positive characteristic via compatibility of cyclotomes

In the present section, we revisit the Grothendieck Conjecture-type results for hyperbolic curves over finitely generated fields of positive characteristic established by Tamagawa, Mochizuki, Stix, and Saïdi-Tamagawa [cf. [20], [26], [27], [28], [29], [30]]. Especially, in the relative situation, we formulate and prove a certain variant of them from the viewpoint of the compatibility of the group-theoretic cyclotomes. In fact, this formulation will be of use in a subsequent joint work with Hoshi and Sawada concerning Grothendieck Conjecture-type results in anabelian geometry.

Let p be a prime number; $\Sigma \subseteq \mathfrak{Primes}$ a nonempty subset of prime numbers that does not contain p; k a field of characteristic p. For each hyperbolic curve X over a field, we shall write $\overline{X}$ for the smooth compactification of X over the base field.

Definition 2.1. Let S be a smooth variety over k. Then we shall write

$$\Delta_S \overset{\text{def}}{=} \pi_1^{\text{ét}}(S_{\overline{k}})^{\Sigma}; \quad \Pi_S \overset{\text{def}}{=} \pi_1^{\text{ét}}(S)/\mathrm{Ker}(\pi_1^{\text{ét}}(S_{\overline{k}}) \twoheadrightarrow \Delta_S),$$

where $\pi_1^{\text{ét}}(S_{\overline{k}}) \twoheadrightarrow \Delta_S$ denotes the natural surjection. In particular, the natural exact sequence of profinite groups

$$1 \longrightarrow \pi_1^{\text{ét}}(S_{\overline{k}}) \longrightarrow \pi_1^{\text{ét}}(S) \longrightarrow G_k \longrightarrow 1$$

[cf. [7], Exposé IX, Théorème 6.1] induces a natural exact sequence of profinite groups

$$1 \longrightarrow \Delta_S \longrightarrow \Pi_S \longrightarrow G_k \longrightarrow 1.$$

Lemma 2.2. *Let X be a hyperbolic curve of genus ≥ 2 over k. Then the following hold:*

(i) The dual $\widehat{\mathbb{Z}}^{\Sigma}$-module of the second cohomology group $H^2(\Delta_{\overline{X}}, \widehat{\mathbb{Z}}^{\Sigma})$ is isomorphic to $\widehat{\mathbb{Z}}^{\Sigma}(1)$ as G_k-modules. Here, "(1)" denotes the Tate twist, i.e., $\widehat{\mathbb{Z}}^{\Sigma}(1) \overset{\text{def}}{=} \varprojlim_n \mu_n(\overline{k})$, where n ranges over the positive integers whose prime factors are contained in Σ.

(ii) Let d be a positive integer;

$$Y \longrightarrow X$$

a finite étale Galois covering tamely ramified along the cusps of X [i.e., points $\in \overline{X} \setminus X$] whose geometric degree [i.e., the degree of the finite morphism $Y_{\overline{k}} \to X_{\overline{k}}$ induced by the covering $Y \to X$] is equal to d. [In particular, Y is a hyperbolic curve of genus ≥ 2 over a finite Galois extension of k.] Write $\widehat{\mathbb{Z}}^{\Sigma}(-1) \overset{\text{def}}{=} \mathrm{Hom}(\widehat{\mathbb{Z}}^{\Sigma}(1), \widehat{\mathbb{Z}}^{\Sigma})$. Then the natural composite homomorphism

$$\widehat{\mathbb{Z}}^{\Sigma}(-1) \xleftarrow{\sim} H^2(\Delta_{\overline{X}}, \widehat{\mathbb{Z}}^{\Sigma}) \longrightarrow H^2(\Delta_{\overline{Y}}, \widehat{\mathbb{Z}}^{\Sigma}) \xrightarrow{\sim} \widehat{\mathbb{Z}}^{\Sigma}(-1)$$

of G_k-modules [cf. (i)] induced by the natural open homomorphism $\Delta_{\overline{Y}} \to \Delta_{\overline{X}}$ is given by multiplication by d. In particular, we have a natural isomorphism

$$H^2(\Delta_{\overline{X}}, \widehat{\mathbb{Z}}^{\Sigma}) \xrightarrow{\sim} \frac{1}{d} \cdot H^2(\Delta_{\overline{X}}, \widehat{\mathbb{Z}}^{\Sigma}) \xrightarrow{\sim} H^2(\Delta_{\overline{Y}}, \widehat{\mathbb{Z}}^{\Sigma})$$

of G_k-modules.

Proof. First, we verify assertion (i). Observe that since $\overline{X}$ is a smooth proper curve of genus ≥ 2, there exists a natural isomorphism

$$H^2(\Delta_{\overline{X}}, \widehat{\mathbb{Z}}^{\Sigma}) \xrightarrow{\sim} H^2_{\text{ét}}(\overline{X}_{\overline{k}}, \widehat{\mathbb{Z}}^{\Sigma})$$

of G_k-modules. On the other hand, it follows immediately from Poincaré duality that $H^2_{\text{ét}}(\overline{X}_{\overline{k}}, \widehat{\mathbb{Z}}^{\Sigma})$ is isomorphic to $\widehat{\mathbb{Z}}^{\Sigma}(-1)$ as G_k-modules. Thus, we conclude that the dual $\widehat{\mathbb{Z}}^{\Sigma}$-module of $H^2(\Delta_{\overline{X}}, \widehat{\mathbb{Z}}^{\Sigma})$ is isomorphic to $\widehat{\mathbb{Z}}^{\Sigma}(1)$ as G_k-modules, as desired. This completes the proof of assertion (i).

Next, we verify assertion (ii). Note that since the geometric degree of the finite morphism $f : \overline{Y} \to \overline{X}$ [induced by the finite étale Galois covering $Y \to X$] is d, it holds that $\deg f^*\mathcal{L} = d \cdot \deg \mathcal{L}$ for each line bundle $\mathcal{L}$ on $\overline{X}$. Then, in light of the construction of Poincaré duality, we observe that the natural composite

$$\widehat{\mathbb{Z}}^{\Sigma}(-1) \xleftarrow{\sim} H^2_{\text{ét}}(\overline{X}_{\overline{k}}, \widehat{\mathbb{Z}}^{\Sigma}) \longrightarrow H^2_{\text{ét}}(\overline{Y}_{\overline{k}}, \widehat{\mathbb{Z}}^{\Sigma}) \xrightarrow{\sim} \widehat{\mathbb{Z}}^{\Sigma}(-1)$$

is given by multiplication by d. This completes the proof of assertion (ii), hence of Lemma 2.2. ☐

Definition 2.3. Let X be a hyperbolic curve over k. Then we shall write

$$\Lambda_X$$

for the dual $\widehat{\mathbb{Z}}^{\Sigma}$-module of the direct limit

$$\varinjlim_{\Pi_Y \subseteq \Pi_X} H^2(\Delta_{\overline{Y}}, \widehat{\mathbb{Z}}^{\Sigma}),$$

where $\Pi_Y \subseteq \Pi_X$ ranges over the normal open subgroups that correspond to the geometrically pro-Σ finite étale Galois coverings $Y \to X$ whose domain curves are of genus ≥ 2 [In particular, $\overline{Y}$ is a smooth proper curve of genus ≥ 2]; the transition maps are isomorphisms that appear in the final display of Lemma 2.2, (ii). In particular, it follows immediately from Lemma 2.2, (i), (ii), that there exists a natural isomorphism

$$\Lambda_X \xrightarrow{\sim} \widehat{\mathbb{Z}}^{\Sigma}(1)$$

of G_k-modules.

Lemma 2.4. *Let X be a hyperbolic curve over k. Suppose that there exists a prime number $l \in \Sigma$ such that the l-adic cyclotomic character $G_k \to \mathbb{Z}_l^{\times}$ associated to k is open. Then the following hold:*

(i) The cuspidal inertia subgroups of Δ_X [i.e., inertia subgroups of Δ_X associated to points $\in \overline{X} \setminus X$] may be reconstructed, in a purely group-theoretic way, from [the underlying topological group structure of] Π_X, together with the normal closed subgroup $\Delta_X \subseteq \Pi_X$.

(ii) The cyclotome Λ_X may be reconstructed, in a purely group-theoretic way, from [the underlying topological group structure of] Π_X, together with the normal closed subgroup $\Delta_X \subseteq \Pi_X$. Here, the output of this group-theoretic reconstruction procedure is an object in the category of profinite groups and is not an object equipped with an isomorphism as in the final display of Definition 2.3.

Proof. First, we verify assertion (i). Note that the image of the outer representation $G_k \to \mathrm{Out}(\Delta_X)$ may be reconstructed from the data $(\Pi_X, \Delta_X \subseteq \Pi_X)$. Thus, we conclude from [the proof of] [19], Corollary 2.7, (i), together with our assumption on the l-adic cyclotomic character associated to k, that the cuspidal inertia subgroups of Δ_X may be reconstructed from the data $(\Pi_X, \Delta_X \subseteq \Pi_X)$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let $Y \to X$ be a geometrically pro-Σ finite tamely ramified Galois covering. Then one may observe that

- the genus of Y may be determined by [the underlying topological group structure of] Δ_Y, together with the cardinality of the set of cusps of Y;
- there exists a natural bijection between the set of cusps of Y and the set of the conjugacy classes of cuspidal inertia subgroups of Δ_Y;
- the kernel of the natural [outer] surjection $\Delta_Y \twoheadrightarrow \Delta_{\overline{Y}}$ is topologically generated by the cuspidal inertia subgroups of Δ_Y;
- every cuspidal inertia subgroup of Δ_Y coincides with the intersection of a cuspidal inertia subgroup of Δ_X with Δ_Y.

Therefore, in light of the construction of Λ_X, assertion (ii) follows immediately from assertion (i). This completes the proof of assertion (ii), hence of Lemma 2.4. □

Definition 2.5.

(i) Let S_1, S_2 be k-schemes. Then we shall write

$$\mathrm{Isom}_k(S_1, S_2)$$

for the set of isomorphisms $S_1 \xrightarrow{\sim} S_2$ of schemes over k.

(ii) Let X_1, X_2 be smooth varieties over k. Then we shall write

$$\mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

for the set of isomorphisms $\Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}$ of profinite groups that lie over the identity automorphism of G_k considered up to composition with an inner automorphism determined by an element of Δ_{X_2} $(\subseteq \Pi_{X_2})$. Suppose that X_1 and X_2 are hyperbolic curves over k. Then we shall write

$$\mathrm{Isom}^{\Lambda}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \subseteq \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

for the subset determined by the isomorphisms $\sigma : \Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}$ [that lie over the identity automorphism of G_k] such that the natural composite isomorphism

$$\widehat{\mathbb{Z}}^{\Sigma}(1) \xleftarrow{\sim} \Lambda_{X_1} \xrightarrow{\sim} \Lambda_{X_2} \xrightarrow{\sim} \widehat{\mathbb{Z}}^{\Sigma}(1)$$

[cf. the final display of Definition 2.3; Lemma 2.4, (ii)] induced by σ is the identity automorphism.

Theorem 2.6. *Suppose that k is a finite field [of characteristic p], and the cardinality of the subset $\mathfrak{Primes} \setminus \Sigma \subseteq \mathfrak{Primes}$ is finite. Let X_1, X_2 be hyperbolic curves over k. Then the natural map*

$$\mathrm{Isom}_k(X_1, X_2) \longrightarrow \mathrm{Isom}^{\Lambda}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

is bijective.

Proof. Let $\sigma : \Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}$ be an isomorphism of profinite groups that lies over the identity automorphism of G_k. Write $\widetilde{X}_1 \to X_1$, $\widetilde{X}_2 \to X_2$ for the respective geometrically pro-Σ universal coverings associated to Π_{X_1}, Π_{X_2}. Then it follows from [27], Theorem D, that σ arises from the following commutative diagram of pro-schemes

$$\begin{array}{ccc} \widetilde{X}_1 & \xrightarrow{\sim} & \widetilde{X}_2 \\ \big\downarrow & & \big\downarrow \\ X_1 & \xrightarrow{\sim} & X_2, \end{array}$$

where the vertical arrows denote the natural morphisms; the horizontal arrows denote the isomorphisms. Here, we observe that this commutative diagram induces an isomorphism $(X_1)_{\overline{k}} \xrightarrow{\sim} (X_2)_{\overline{k}}$ of schemes and an automorphism $\overline{k} \xrightarrow{\sim} \overline{k}$ of fields. On the other hand, since the cardinality of the subset $\mathfrak{Primes} \setminus \Sigma \subseteq \mathfrak{Primes}$ is finite, it follows from [8], Theorem A, that the natural homomorphism $G_{\mathbb{F}_p} \to \mathrm{Aut}(\widehat{\mathbb{Z}}^{\Sigma}(1))$ is injective. Then it follows immediately from the various definitions involved that the element $\in G_{\mathbb{F}_p} \hookrightarrow \mathrm{Aut}(\widehat{\mathbb{Z}}^{\Sigma}(1))$ determined by the automorphism $\overline{k} \xrightarrow{\sim} \overline{k}$ coincides with the inverse of the element $\in \mathrm{Aut}(\widehat{\mathbb{Z}}^{\Sigma}(1))$ that corresponds to the natural composite isomorphism $\widehat{\mathbb{Z}}^{\Sigma}(1) \xleftarrow{\sim} \Lambda_{X_1} \xrightarrow{\sim} \Lambda_{X_2} \xrightarrow{\sim} \widehat{\mathbb{Z}}^{\Sigma}(1)$ [cf. the final display of Definition 2.3; Lemma 2.4, (ii)]. In particular, if the natural composite isomorphism $\widehat{\mathbb{Z}}^{\Sigma}(1) \xleftarrow{\sim} \Lambda_{X_1} \xrightarrow{\sim} \Lambda_{X_2} \xrightarrow{\sim} \widehat{\mathbb{Z}}^{\Sigma}(1)$ is the identity automorphism, then the isomorphism $(X_1)_{\overline{k}} \xrightarrow{\sim} (X_2)_{\overline{k}}$ is an isomorphism of $\overline{k}$-schemes. These observations immediately imply that the natural map

$$\mathrm{Isom}_k(X_1, X_2) \longrightarrow \mathrm{Isom}^{\Lambda}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

is bijective, as desired. This completes the proof of Theorem 2.6. $\square$

Remark 2.6.1. The condition that the subset $\mathfrak{Primes} \setminus \Sigma \subseteq \mathfrak{Primes}$ is finite that appears in Theorem 2.6 [and in the results of the present paper] may be replaced by a weaker condition as in [27], Theorem D. On the other hand, since the condition in [27], Theorem D depends on respective hyperbolic curves under considerations, the author has decided to assume a stronger condition throughout the present paper.

Remark 2.6.2. Note that the absolute Galois groups of finite fields are abelian. In particular, the relative Grothendieck Conjecture for hyperbolic curves over finite fields in the usual formulation does not hold. This situation has made the author to introduce the modified formulation as in Theorem 2.6. Moreover, in the case where we consider Grothendieck Conjecture for hyperbolic curves over more general base fields, if we restrict our attention to such a modified formulation, then we do not need to treat the isotrivial and nonisotrivial cases separately [cf. Theorems 2.9, 4.5 below].

Lemma 2.7. *Let S be a connected regular scheme of characteristic p; $s \in S$ a closed point; $\mathcal{X} \to S$ a proper hyperbolic curve. Write $\eta \in S$ for the generic point of S; $\mathcal{X}_\eta \overset{\mathrm{def}}{=} \mathcal{X} \times_S \eta$; $\mathcal{X}_s \overset{\mathrm{def}}{=} \mathcal{X} \times_S s$. Then the natural composite isomorphism*

$$\widehat{\mathbb{Z}}^{\Sigma}(1) \xleftarrow{\sim} \Lambda_{\mathcal{X}_\eta} \xrightarrow{\sim} \Lambda_{\mathcal{X}_s} \xrightarrow{\sim} \widehat{\mathbb{Z}}^{\Sigma}(1)$$

induced by the specialization isomorphism $\Delta_{\mathcal{X}_\eta} \xrightarrow{\sim} \Delta_{\mathcal{X}_s}$ [cf. also the final display of Definition 2.3] is the identity automorphism.

Proof. Write $\mathcal{X}_{\overline{\eta}}$, $\mathcal{X}_{\overline{s}}$ for the geometric fibers of $\mathcal{X} \to S$ over η, s, respectively. Then we have the following commutative diagram:

$$\begin{array}{ccc} H^2(\Delta_{\mathcal{X}_s}, \widehat{\mathbb{Z}}^{\Sigma}) & \xrightarrow{\sim} & H^2_{\text{ét}}(\mathcal{X}_{\overline{s}}, \widehat{\mathbb{Z}}^{\Sigma}) \\ \big\downarrow \wr & & \big\downarrow \wr \\ H^2(\Delta_{\mathcal{X}_\eta}, \widehat{\mathbb{Z}}^{\Sigma}) & \xrightarrow{\sim} & H^2_{\text{ét}}(\mathcal{X}_{\overline{\eta}}, \widehat{\mathbb{Z}}^{\Sigma}), \end{array}$$

where the horizontal arrows denote the natural isomorphisms [cf. our assumption that $\mathcal{X} \to S$ is a proper hyperbolic curve]; the left-hand vertical arrow denotes the isomorphism induced by the specialization isomorphism $\Delta_{\mathcal{X}_\eta} \xrightarrow{\sim} \Delta_{\mathcal{X}_s}$; the right-hand vertical arrow denotes the specialization isomorphism. On the other hand, we observe that the trace maps $H^2_{\text{ét}}(\mathcal{X}_{\overline{s}}, \widehat{\mathbb{Z}}^\Sigma(1)) \xrightarrow{\sim} \widehat{\mathbb{Z}}^\Sigma$ and $H^2_{\text{ét}}(\mathcal{X}_{\overline{\eta}}, \widehat{\mathbb{Z}}^\Sigma(1)) \xrightarrow{\sim} \widehat{\mathbb{Z}}^\Sigma$ [induced by the long cohomology exact sequences associated to Kummer exact sequences] are compatible with the specialization isomorphism $H^2_{\text{ét}}(\mathcal{X}_{\overline{s}}, \widehat{\mathbb{Z}}^\Sigma(1)) \xrightarrow{\sim} H^2_{\text{ét}}(\mathcal{X}_{\overline{\eta}}, \widehat{\mathbb{Z}}^\Sigma(1))$. Thus, in light of the construction of Poincaré duality, we conclude that the natural composite isomorphism $\widehat{\mathbb{Z}}^\Sigma(1) \xleftarrow{\sim} \Lambda_{\mathcal{X}_\eta} \xrightarrow{\sim} \Lambda_{\mathcal{X}_s} \xrightarrow{\sim} \widehat{\mathbb{Z}}^\Sigma(1)$ is the identity automorphism. This completes the proof of Lemma 2.7. □

Lemma 2.8. *Let S be an algebraic variety over a field; Z a scheme; $\phi : Z \to S$ a finite unramified morphism of schemes. Suppose that, for any closed point $s \in S$, the closed immersion* Spec $k(s) \to S$ *[where $k(s)$ denotes the residue field of the local ring $\mathcal{O}_{S,s}$ at $s \in S$] lifts to a morphism* Spec $k(s) \to Z$ *via ϕ. Write $\eta \in S$ for the generic point of S. Then ϕ is totally split in an étale neighborhood of $\eta \in S$.*

Proof. Note that it follows immediately from our assumption that the image of ϕ contains the subset of closed points of S. On the other hand, since S is an algebraic variety over a field, the subset of closed points of S is dense. Therefore, the finiteness of ϕ implies that ϕ is surjective. Recall that the étale locus of ϕ is open. In particular, since ϕ is a finite unramified morphism, by replacing S by an open subscheme of S, we may assume without loss of generality that ϕ is a finite étale, and surjective morphism. Thus, we conclude that there exists a suitable finite étale Galois covering $W \to S$ such that the morphism $W \times_S Z \to W$ is totally split, as desired. This completes the proof of Lemma 2.8. □

Theorem 2.9. *Suppose that k is a finitely generated field [of characteristic p], and the cardinality of the subset $\mathfrak{Primes} \setminus \Sigma \subseteq \mathfrak{Primes}$ is finite. Let X_1, X_2 be hyperbolic curves over k. Then the natural map*

$$\mathrm{Isom}_k(X_1, X_2) \longrightarrow \mathrm{Isom}^\Lambda_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

is bijective.

Proof. First, the injectivity of the natural map in question is well-known. In the remainder, we discuss the surjectivity of the natural map. Recall that every almost pro-Σ surface group is slim [cf. [22], Proposition 1.4]. Then, in light of Galois descent, together with Hurwitz's formula, by replacing X_1, X_2, by suitable respective geometrically pro-Σ finite étale Galois coverings, we may assume without loss of generality that X_1 and X_2 have genus ≥ 2. On the other hand, since k is a finitely generated field, it follows immediately from Lemma 2.4, (i), that every isomorphism $\Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}$ of profinite groups that lies over the identity automorphism of G_k induces a bijection between the respective sets of cuspidal inertia subgroups of Δ_{X_1}, Δ_{X_2}. Therefore, again by applying a similar argument to the above argument concerning Galois descent, together with the slimness of almost pro-Σ surface groups, we may assume without loss of generality that X_1 and X_2 are proper hyperbolic curves.

Next, let S be a smooth variety over a finite field such that

- the function field of S coincides with k, and
- the proper hyperbolic curves X_1, X_2 over k extend to proper hyperbolic curves $\mathcal{X}_1$, $\mathcal{X}_2$ over S, respectively.

Write

$$\mathbf{Isom}_S(\mathcal{X}_1, \mathcal{X}_2)$$

for the Isom scheme over S determined by $\mathcal{X}_1$ and $\mathcal{X}_2$. Recall that $\mathbf{Isom}_S(\mathcal{X}_1, \mathcal{X}_2)$ is finite and unramified over S [cf. [2], Theorem 1.11]. On the other hand, let $s \in S$ be a closed point; $\sigma : \Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}$ an isomorphism of profinite groups that determines an element $\in \mathrm{Isom}^{\Lambda}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$. Write $(\mathcal{X}_1)_s \stackrel{\mathrm{def}}{=} \mathcal{X}_1 \times_S s$; $(\mathcal{X}_2)_s \stackrel{\mathrm{def}}{=} \mathcal{X}_2 \times_S s$; $k(s)$ for the residue field of S at s. Then since $\mathcal{X}_1$ and $\mathcal{X}_2$ are proper hyperbolic curves over S, it follows immediately from Lemma 2.7 that σ determines an element $\sigma_s \in \mathrm{Isom}^{\Lambda}_{G_{k(s)}}(\Pi_{(\mathcal{X}_1)_s}, \Pi_{(\mathcal{X}_2)_s})/\mathrm{Inn}(\Delta_{(\mathcal{X}_2)_s})$. Therefore, by applying Theorem 2.6, we observe that σ_s arises from a unique isomorphism $(\mathcal{X}_1)_s \xrightarrow{\sim} (\mathcal{X}_2)_s$ over $k(s)$. In particular, the structure morphism $\mathbf{Isom}_S(\mathcal{X}_1, \mathcal{X}_2) \to S$ satisfies the assumption of Lemma 2.8. Here, we note that, in light of Galois descent, to verify that σ arises from a(n) [unique] isomorphism $X_1 \xrightarrow{\sim} X_2$ over k, one may replace k by a finite Galois extension of k. Thus, in light of Lemma 2.8, by replacing k by a finite Galois extension of k, S by an étale locus of S, and $s \in S$ by a suitable closed point of S, we may assume without loss of generality that there exists an isomorphism $\mathcal{X}_1 \xrightarrow{\sim} \mathcal{X}_2$ over S that lifts the isomorphism $(\mathcal{X}_1)_s \xrightarrow{\sim} (\mathcal{X}_2)_s$ over $k(s)$. Thus, we conclude from the theory of specialization that the isomorphism $X_1 \xrightarrow{\sim} X_2$ over k induced by the isomorphism $\mathcal{X}_1 \xrightarrow{\sim} \mathcal{X}_2$ over S maps to σ via the natural map $\mathrm{Isom}_k(X_1, X_2) \longrightarrow \mathrm{Isom}^{\Lambda}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$. This completes the proof of Theorem 2.9. $\square$

Remark 2.9.1. The argument applied in the proof of Theorem 2.9 is similar to the argument applied in [30], §6.

Definition 2.10. Let X be a hyperbolic curve over k; $\phi : \pi_1^{\mathrm{tame}}(X) \twoheadrightarrow \Pi$ a quotient of profinite groups such that the natural quotient $\pi_1^{\mathrm{tame}}(X) \twoheadrightarrow G_k$ factors through ϕ. Write Δ for the kernel of the natural surjection $\Pi \twoheadrightarrow G_k$. Then we shall say that Π is Σ-*closed* if, for each normal open subgroup $N \subseteq \Pi$, the natural surjection $(\phi^{-1}(N) \cap \pi_1^{\mathrm{tame}}(X_{\overline{k}}))^{\Sigma} \twoheadrightarrow (N \cap \Delta)^{\Sigma}$ is an isomorphism.

Definition 2.11. Let X be a hyperbolic curve over k; $N \subseteq \pi_1^{\mathrm{tame}}(X)$ a normal open subgroup. Then we shall refer to the quotient

$$\pi_1^{\mathrm{tame}}(X)/\mathrm{Ker}(N \cap \pi_1^{\mathrm{tame}}(X_{\overline{k}}) \twoheadrightarrow (N \cap \pi_1^{\mathrm{tame}}(X_{\overline{k}}))^{\Sigma})$$

as an *almost geometrically pro-*Σ *quotient* of $\pi_1^{\mathrm{tame}}(X)$ associated to N.

Remark 2.11.1. Let X be a hyperbolic curve over k. Then it follows immediately from the various definitions involved that $\pi_1^{\mathrm{tame}}(X)$ and any almost geometrically pro-Σ quotient of $\pi_1^{\mathrm{tame}}(X)$ are Σ-closed. In particular, Π_X is Σ-closed. Moreover, every Σ-closed quotient of $\pi_1^{\mathrm{tame}}(X)$ may be identified with the inverse limit of an inverse system consisting of almost geometrically pro-Σ quotients of $\pi_1^{\mathrm{tame}}(X)$.

Lemma 2.12. *Let X be a hyperbolic curve over k; Π a Σ-closed quotient of $\pi_1^{\mathrm{tame}}(X)$; $Y \to X$ a finite étale Galois covering tamely ramified along the cusps of X that corresponds to a normal open subgroup $\Pi^Y \subseteq \Pi$. Then one may reconstruct the natural isomorphism $\Lambda_Y \xrightarrow{\sim} \Lambda_X$ [compatible with the natural scheme-theoretic identifications $\Lambda_X \xrightarrow{\sim} \widehat{\mathbb{Z}}^{\Sigma}(1)$, $\Lambda_Y \xrightarrow{\sim} \widehat{\mathbb{Z}}^{\Sigma}(1)$], in a purely group-theoretic way, from [the underlying topological group structure of] Π, together the normal closed subgroups $\Pi^Y \subseteq \Pi$, $\mathrm{Ker}(\Pi \twoheadrightarrow G_k) \subseteq \Pi$.*

Proof. Lemma 2.12 follows immediately from Lemma 2.2, (ii), together with the various definitions involved. $\square$

Definition 2.13. Let X be a hyperbolic curve over k; Π a Σ-closed quotient of $\pi_1^{\mathrm{tame}}(X)$. For each normal open subgroup $N \subseteq \Pi$, write $X_N \to X$ for the finite tamely ramified Galois covering associated to $N \subseteq \Pi$. Then we shall write

$$\Lambda_\Pi \stackrel{\mathrm{def}}{=} \varinjlim_{N \subseteq \Pi} \Lambda_{X_N},$$

where $N \subseteq \Pi$ ranges over the normal open subgroups; the transition maps are the isomorphisms as in Lemma 2.12. In particular, we have a natural isomorphism $\Lambda_\Pi \xrightarrow{\sim} \widehat{\mathbb{Z}}^\Sigma(1)$.

Remark 2.13.1. In the notation of Definition 2.13, suppose that there exists a prime number $l \in \Sigma$ such that the l-adic cyclotomic character $G_k \to \mathbb{Z}_l^\times$ associated to k is open. Then it follows immediately from Lemma 2.4, (ii), together with the various definitions involved, that Λ_Π may be reconstructed, in a purely group-theoretic way, from [the underlying topological group structure of] Π.

Corollary 2.14. *Let X_1, X_2 be hyperbolic curves over k;*

$$\Pi_1, \ \Pi_2$$

Σ-closed quotients of $\pi_1^{\mathrm{tame}}(X_1)$, $\pi_1^{\mathrm{tame}}(X_2)$, respectively;

$$\sigma : \Pi_1 \xrightarrow{\sim} \Pi_2$$

a $\mathrm{Ker}(\Pi_2 \twoheadrightarrow G_k)$*-outer isomorphism of profinite groups that lies over the identity automorphism of G_k. Suppose that*

- *k is a finitely generated field [of characteristic p],*
- *the cardinality of the subset $\mathfrak{Primes} \setminus \Sigma \subseteq \mathfrak{Primes}$ is finite, and*
- *the composite isomorphism $\widehat{\mathbb{Z}}^\Sigma(1) \xleftarrow{\sim} \Lambda_{\Pi_1} \xrightarrow{\sim} \Lambda_{\Pi_2} \xrightarrow{\sim} \widehat{\mathbb{Z}}^\Sigma(1)$ induced by σ [cf. Remark 2.13.1] is the identity automorphism.*

Then σ arises from a unique isomorphism

$$X_1 \xrightarrow{\sim} X_2$$

over k.

Proof. Note that since Π_1 and Π_2 are Σ-closed quotients, Π_1 and Π_2 may be identified with the respective inverse limits of inverse systems consisting of almost geometrically pro-Σ quotients of $\pi_1^{\mathrm{tame}}(X_1)$, $\pi_1^{\mathrm{tame}}(X_2)$ compatible with σ [cf. Remark 2.11.1]. Thus, we may assume without loss of generality that Π_1, Π_2 coincide with almost geometrically pro-Σ quotients of $\pi_1^{\mathrm{tame}}(X_1)$, $\pi_1^{\mathrm{tame}}(X_2)$, respectively. In this situation, in light of Galois descent, together with the slimness of almost pro-Σ surface groups [cf. [22], Proposition 1.4], we conclude from Theorem 2.9 that σ arises from a unique isomorphism $X_1 \xrightarrow{\sim} X_2$ over k, as desired. This completes the proof of Corollary 2.14. $\square$

3 Automorphisms of the perfections of positive characteristic discrete valuation fields that induce the identity outer automorphisms of the absolute Galois groups

Let p be a prime number; k a discrete valuation field of characteristic p; $\sigma \in \mathrm{Aut}(k^{\mathrm{pf}})$. Write

$$\phi : \mathrm{Aut}(k^{\mathrm{pf}}) \longrightarrow \mathrm{Out}(G_{k^{\mathrm{pf}}})$$

for the natural homomorphism [cf. [10], Introduction]. In the present section, we prove that if $\phi(\sigma)$ is trivial, then there exists a unique integer m such that σ coincides with the p^m-th Frobenius automorphism of k^{pf}. The proof may be regarded as a certain modified version of the discussion applied in [10], i.e., a combination of Kummer theory and Artin-Schreier theory with respect to cyclotomes. This result may be applied to give an enhanced version of a well-known Pop's theorem in birational anabelian geometry in the next section.

For each field F of characteristic p and each positive integer n, we shall write

$$\wp_{p^n} : F \longrightarrow F$$

for the Artin-Schreier map that maps $F \ni x \mapsto x^{p^n} - x \in F$;

$$\wp_{p^\infty}(F) \stackrel{\mathrm{def}}{=} \bigcap_{m \geq 1} \mathrm{Im}(\wp_{p^m}) \subseteq F,$$

where m ranges over the positive integers. Write $\mathbb{Z}[\frac{1}{p^\infty}] \subseteq \mathbb{Q}$ for the [additive] subgroup generated by the negative integer powers of p;

$$\mathcal{O}_{k^{\mathrm{pf}}} \subseteq k^{\mathrm{pf}}$$

for the ring of integers of k^{pf}.

Lemma 3.1. *Let F be a field of characteristic p that admits a surjective homomorphism $F^\times \twoheadrightarrow \mathbb{Z}[\frac{1}{p^\infty}]$;*

$$\tau \in \mathrm{Aut}(F)$$

an automorphism that induces the identity outer automorphism of G_F. Fix a lifting

$$\tilde{\tau} \in \mathrm{Aut}(F^{\mathrm{sep}})$$

of τ that induces the identity automorphism of G_F [cf. [10], Lemma 1.2]. Then the following hold:

(i) Write $c \in (\widehat{\mathbb{Z}}^{(p)'})^\times = \mathrm{Aut}(\widehat{\mathbb{Z}}^{(p)'}(1))$ for the element determined by $\tilde{\tau}$. Then there exists a unique integer m such that $c = p^m$. In particular, $\tilde{\tau}$ induces the p^m-th Frobenius automorphism on $\overline{\mathbb{F}}_p$ ($\subseteq F^{\mathrm{sep}}$).

(ii) Suppose that F is perfect. Then the automorphism $F/\wp_{p^\infty}(F) \xrightarrow{\sim} F/\wp_{p^\infty}(F)$ of $\mathbb{F}_p$-vector space induced by τ coincides with the automorphism induced by the p^m-th Frobenius automorphism of F, where m denotes the integer that appears in (i).

Proof. First, we verify assertion (i). Write

$$\kappa_F : F^\times \longrightarrow H^1(G_F, \widehat{\mathbb{Z}}^{(p)'}(1))$$

for the Kummer map. Note that it follows immediately from the various definitions involved that κ_F is compatible with $\tau : F \xrightarrow{\sim} F$ and the automorphism of $H^1(G_F, \widehat{\mathbb{Z}}^{(p)'}(1))$ induced by $\tilde{\tau}$. On the other hand, since $\tilde{\tau}$ induces the identity automorphism of G_F, it follows from the definition of c that the automorphism of $H^1(G_F, \widehat{\mathbb{Z}}^{(p)'}(1))$ induced by $\tilde{\tau}$ is given by multiplication by c. Then the existence of a surjective homomorphism $F^\times \twoheadrightarrow \mathbb{Z}[\frac{1}{p^\infty}]$ implies that the automorphism of $\widehat{\mathbb{Z}}^{(p)'}$ given by multiplication by c preserves the subgroup $\mathbb{Z}[\frac{1}{p^\infty}] \subseteq \widehat{\mathbb{Z}}^{(p)'}$. Here, we observe that $(\widehat{\mathbb{Z}}^{(p)'})^\times \cap \mathbb{Q} = \{\pm p^n\}_{n\in\mathbb{Z}}$. In particular, it holds that $c \in \{\pm p^n\}_{n\in\mathbb{Z}}$. Note that the field automorphism of $\overline{\mathbb{F}}_p$ induced by $\tilde{\tau}$ is given by the assignment $\overline{\mathbb{F}}_p^\times \ni x \mapsto x^c \in \overline{\mathbb{F}}_p^\times$. Thus, since the assignment $\overline{\mathbb{F}}_p^\times \ni x \mapsto x^{-1} \in \overline{\mathbb{F}}_p^\times$ does not give rise to a field automorphism of $\overline{\mathbb{F}}_p$, we conclude that $c \in \{p^n\}_{n\in\mathbb{Z}}$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Since F is perfect, by replacing τ by the composite of τ with the p^{-m}-th Frobenius automorphism, we may assume without loss of generality that $m = 0$. In particular, it holds that $c = 1$. In this situation, it suffices to verify that, for each positive integer n, $\tilde{\tau}$ induces the identity automorphism of $F/\mathrm{Im}(\wp_{p^n})$. Observe that there exists a natural isomorphism

$$F/\mathrm{Im}(\wp_{p^n}) \xrightarrow{\sim} H^1(G_F, \mathbb{F}_{p^n})$$

that arises from the natural exact sequence

$$1 \longrightarrow \mathbb{F}_{p^n} \longrightarrow F^{\mathrm{sep}} \xrightarrow{\wp_{p^n}} F^{\mathrm{sep}} \longrightarrow 1.$$

Thus, since $\tilde{\tau}$ induces the identity automorphism of G_F, we conclude from assertion (i) that $\tilde{\tau}$ induces the identity automorphism of $F/\mathrm{Im}(\wp_{p^n})$. This completes the proof of assertion (ii), hence of Lemma 3.1. $\square$

Lemma 3.2. *It holds that*

$$\wp_{p^\infty}(k^{\mathrm{pf}}) \subseteq \mathcal{O}_{k^{\mathrm{pf}}}.$$

Proof. Let $a \in \wp_{p^\infty}(k^{\mathrm{pf}}) \setminus \mathcal{O}_{k^{\mathrm{pf}}}$ be an element. By replacing k by a finite [purely inseparable] extension field of k, we may assume without loss of generality that $a \in k$. Observe that, for each positive integer n, the finite field extension of k obtained by adjoining the roots of the equation $x^{p^n} - x - a = 0$ is separable. This observation implies that $a \in \wp_{p^\infty}(k)$. Thus, we conclude from [10], Lemma 2.2, (iii), that $a \in \mathcal{O}_{k^{\mathrm{pf}}}$. This completes the proof of Lemma 3.2. $\square$

Lemma 3.3. *Suppose that $\phi(\sigma)$ is trivial. Then σ preserves the subset $k^{\mathrm{pf}} \setminus \mathcal{O}_{k^{\mathrm{pf}}} \subseteq k^{\mathrm{pf}}$. Moreover, there exists a unique integer m such that the automorphism of $k^{\mathrm{pf}} \setminus \mathcal{O}_{k^{\mathrm{pf}}}$ induced by σ coincides with the automorphism determined by the assignment $k^{\mathrm{pf}} \setminus \mathcal{O}_{k^{\mathrm{pf}}} \ni x \mapsto x^{p^m} \in k^{\mathrm{pf}} \setminus \mathcal{O}_{k^{\mathrm{pf}}}$.*

Proof. Note that the discrete valuation on k gives a surjective homomorphism $(k^{\mathrm{pf}})^\times \twoheadrightarrow \mathbb{Z}[\frac{1}{p^\infty}]$. Then it follows immediately from Lemma 3.1, (ii), that, by replacing σ by the composite of σ with a suitable Frobenius automorphism, we may assume without loss of generality that σ induces the identity automorphism on $k^{\mathrm{pf}}/\wp_{p^\infty}(k^{\mathrm{pf}})$. Thus, in light of Lemma 3.2, we conclude from a similar argument to the argument applied in the proof of [10], Lemma 2.3, that σ is trivial. This completes the proof of Lemma 3.3. $\square$

Theorem 3.4. *Suppose that $\phi(\sigma)$ is trivial. Then there exists a unique integer m such that σ coincides with the p^m-th Frobenius automorphism.*

Proof. Theorem 3.4 follows immediately from Lemma 3.3, together with the fact that every element $\in \mathcal{O}_{k^{\mathrm{pf}}}$ may be written as the sum of two elements $\in k^{\mathrm{pf}} \setminus \mathcal{O}_{k^{\mathrm{pf}}}$. $\square$

4 Absolute Grothendieck Conjecture for nonisotrivial hyperbolic curves over the perfections of finitely generated fields of positive characteristic

Throughout the present section, we maintain the notation of §2 and suppose that

the cardinality of the subset $\mathfrak{Primes} \setminus \Sigma \subseteq \mathfrak{Primes}$ is *finite*.

Moreover, we shall say that

> a hyperbolic curve C over the field k [of characteristic p] is *isotrivial* if C is isotrivial relative to the subfield $k \cap \overline{\mathbb{F}}_p \subseteq k$.

In the present section, in light of the isotriviality criterion in §1, together with

- the relative version of the Grothendieck Conjecture formulated in §2, and
- an elementary observation [cf. [9], Theorem 1.2]:

> Let F be an algebraically closed field of characteristic p. Then the image of the graph
>
> $$\mathbb{A}^1(F) \longrightarrow \mathbb{A}^1(F) \times \mathbb{A}^1(F)$$
>
> associated to an automorphism $\alpha \in \mathrm{Aut}(F)$ is not Zariski-dense if and only if α is an integral power of the p-th Frobenius automorphism,

we first discuss the partial compatibility of the group-theoretic cyclotomes [cf. Proposition 4.4]. As a direct consequence of this partial compatibility, we give an alternative proof of Stix's results concerning the relative version of the Grothendieck Conjecture for nonisotrivial hyperbolic curves over the finitely generated fields of positive characteristic [cf. Theorem 4.5]. Moreover, by combining with one of the Pop's results in birational anabelian geometry, together with the result obtained in §3, we prove an absolute version of the Grothendieck Conjecture for the geometrically pro-Σ fundamental groups of nonisotrivial hyperbolic curves over the perfections of finitely generated fields of positive characteristic [cf. Theorem 4.10]. This result may be regarded as a higher dimensional base field analogue of [26], Theorem 1; [27], Theorem D.

First, we begin by recalling an elementary property of the relative Frobenius morphisms.

Definition 4.1. Let m be a positive integer; X a hyperbolic curve over k. Write

$$X^{(p^m)}$$

for the hyperbolic curve over k obtained by forming the base extension of X via the morphism Spec $k \to$ Spec k that corresponds to the p^m-th Frobenius endomorphism $k \to k$;

$$F_X^{(p^m)} : X \longrightarrow X^{(p^m)}$$

for the relative p^m-th Frobenius morphism.

Lemma 4.2. *Let m be a positive integer; X a hyperbolic curve over k. Then the composite isomorphism*

$$\widehat{\mathbb{Z}}^{\Sigma}(1) \stackrel{\sim}{\leftarrow} \Lambda_X \stackrel{\sim}{\to} \Lambda_{X^{(p^m)}} \stackrel{\sim}{\to} \widehat{\mathbb{Z}}^{\Sigma}(1)$$

induced by $F_X^{(p^m)}$ is given by multiplication by p^m.

Proof. Lemma 4.2 follows immediately from the construction of Poincaré duality, together with the various definitions involved. $\square$

Lemma 4.3. *Suppose that k is perfect. Let T be an algebraic variety over k of dimension ≥ 1; $\alpha \in G_k$. Write $\Delta \subseteq T \times_k T$ for the diagonal divisor; $W \stackrel{\text{def}}{=} (\mathrm{id} \times \alpha)(\Delta(\overline{k})) \subseteq T(\overline{k}) \times T(\overline{k})$. Then, if $W \subseteq T(\overline{k}) \times T(\overline{k})$ is a constructible subset, then α is an integral power of the p-th Frobenius automorphism.*

Proof. First, by replacing T by an open subvariety of T, if necessary, we may assume without loss of generality that there exists a dominant morphism $\phi : T \to \mathbb{A}^1_k$ over k. Then since W is constructible, it follows from Chevalley's theorem that the subset $(\phi \times \phi)(W) \subseteq \mathbb{A}^1_k(\overline{k}) \times \mathbb{A}^1_k(\overline{k})$ is also constructible. Observe that it follows immediately from the various definitions involved that $(\phi \times \phi)(W) \subseteq (\mathrm{id} \times \alpha)(\Delta_{\mathbb{A}^1}(\overline{k}))$, and the complement $(\mathrm{id} \times \alpha)(\Delta_{\mathbb{A}^1}(\overline{k})) \setminus (\phi \times \phi)(W)$ is finite. Then $(\mathrm{id} \times \alpha)(\Delta_{\mathbb{A}^1}(\overline{k}))$ is also constructible. Here, we note that a constructible subset $C \subseteq \mathbb{A}^2_k(\overline{k})$ is Zariski-dense if and only if C contains a nonempty open subset. On the other hand, if we write $\Delta_{\mathbb{A}^1} \subseteq \mathbb{A}^1_k \times_k \mathbb{A}^1_k$ for the diagonal divisor, then one may observe that the subset $(\mathrm{id} \times \alpha)(\Delta_{\mathbb{A}^1}(\overline{k})) \subseteq \mathbb{A}^1_k(\overline{k}) \times \mathbb{A}^1_k(\overline{k})$ does not contain any nonempty open subset. Therefore, the subset $(\mathrm{id} \times \alpha)(\Delta_{\mathbb{A}^1}(\overline{k})) \subseteq \mathbb{A}^1_k(\overline{k}) \times \mathbb{A}^1_k(\overline{k})$ is not Zariski-dense. Thus, we conclude from [9], Theorem 1.2, that α is an integral power of the p-th Frobenius automorphism. This completes the proof of Lemma 4.3. $\square$

Proposition 4.4. *Suppose that k is a finitely generated transcendental extension field of the prime field [of characteristic p]. Let X_1 be a nonisotrivial hyperbolic curve over k; X_2 a hyperbolic curve over k;*

$$\Pi_1, \ \Pi_2$$

Σ-closed quotients of $\pi_1^{\mathrm{tame}}(X_1)$, $\pi_1^{\mathrm{tame}}(X_2)$, respectively;

$$\sigma : \Pi_1 \xrightarrow{\sim} \Pi_2$$

a $\Delta_2 \stackrel{\text{def}}{=} \mathrm{Ker}(\Pi_2 \twoheadrightarrow G_k)$-outer isomorphism of profinite groups that lies over the identity automorphism of G_k. Then X_2 is also nonisotrivial, and the composite isomorphism

$$\widehat{\mathbb{Z}}^{\Sigma}(1) \xleftarrow{\sim} \Lambda_{\Pi_1} \xrightarrow{\sim} \Lambda_{\Pi_2} \xrightarrow{\sim} \widehat{\mathbb{Z}}^{\Sigma}(1)$$

induced by σ [cf. Remark 2.13.1] is given by multiplication by p^m for some integer m.

Proof. First, it follows immediately from Remark 2.11.1 that we may assume without loss of generality that

$$\Pi_1 = \Pi_{X_1}, \quad \Pi_2 = \Pi_{X_2}.$$

In particular, σ induces an isomorphism between the pro-Σ outer representations associated to X_1, X_2. Then since X_1 is nonisotrivial, it follows immediately from Theorem 1.14 that X_2 is also nonisotrivial.

In the remainder of the present proof, we verify the second assertion. In light of Lemmas 1.13; 2.4, (i); 2.12, by replacing X_1, X_2 by the respective smooth compactifications of suitable geometrically pro-Σ finite étale Galois coverings of X_1, X_2, we may assume without loss of generality that

$$X_1 \text{ and } X_2 \text{ are nonisotrivial proper hyperbolic curves over } k.$$

Next, write

$$\sigma^{\mathrm{cyc}} \in (\widehat{\mathbb{Z}}^{\Sigma})^{\times} = \mathrm{Aut}(\widehat{\mathbb{Z}}^{\Sigma}(1))$$

for the element determined by the composite isomorphism $\widehat{\mathbb{Z}}^{\Sigma}(1) \xleftarrow{\sim} \Lambda_{\Pi_1} \xrightarrow{\sim} \Lambda_{\Pi_2} \xrightarrow{\sim} \widehat{\mathbb{Z}}^{\Sigma}(1)$. Let S be a smooth variety over a finite field such that

- the function field of S coincides with k, and
- the proper hyperbolic curves X_1, X_2 over k extend to proper hyperbolic curves $\mathcal{X}_1$, $\mathcal{X}_2$ over S, respectively.

Then, in light of the theory of specialization, it follows immediately from [27], Theorem D, together with Lemma 2.7, that σ^{cyc} is contained in the image of the natural injection $G_{\mathbb{F}_p} \hookrightarrow \mathrm{Aut}(\widehat{\mathbb{Z}}^{\Sigma}(1))$, and that, for each closed point $s \in S_{\overline{\mathbb{F}}_p}$, σ induces an isomorphism $(\mathcal{X}_1)_s \xrightarrow{\sim} (\mathcal{X}_2)_s$ of schemes that lies over the automorphism $\overline{\sigma} : \overline{\mathbb{F}}_p \xrightarrow{\sim} \overline{\mathbb{F}}_p$ induced by σ^{cyc}. Write $\mathbb{F}_q$ for the field of constant of S. Then, in light of Lemma 4.2, by replacing X_1 or X_2 by a suitable Frobenius twist, we may assume without loss of generality that $\overline{\sigma}$ induces the identity automorphism on $\mathbb{F}_q$. Write g for the genus of X_1 [so the genus of X_2 is also g]; M_g for the coarse moduli scheme of proper hyperbolic curves of genus g over $\mathbb{F}_q$. For each $i = 1, 2$, write

$$f_i : S \longrightarrow M_g$$

for the classifying morphism determined by $\mathcal{X}_i$. Then, in the level of $\overline{\mathbb{F}}_p$-valued points, it holds that $f_2 = \overline{\sigma} \circ f_1$, where we identify $\overline{\sigma}$ with the natural action of $\overline{\sigma}$ on $M_g(\overline{\mathbb{F}}_p)$. On the other hand, by replacing S by an open subvariety of S, we may assume without loss of generality that the image $f_1(S(\overline{\mathbb{F}}_p))(= f_2(S(\overline{\mathbb{F}}_p))$ admits a structure of algebraic subvariety of M_g. Write T for this subvariety; $\Delta \subsetneq T \times_{\mathbb{F}_q} T$ for the diagonal divisor; $(f_1, f_2) : S \to T \times_{\mathbb{F}_q} T$ for the morphism over $\mathbb{F}_q$ determined by f_1 and f_2. Note that since X_1 is nonisotrivial, it holds that T is of dimension ≥ 1. On the other hand, it follows immediately from the various definitions involved that $(\mathrm{id} \times \overline{\sigma})(\Delta(\overline{\mathbb{F}}_p)) = (f_1, f_2)(S(\overline{\mathbb{F}}_p))$. Thus, since $(f_1, f_2)(S(\overline{\mathbb{F}}_p)) \subseteq T(\overline{\mathbb{F}}_p) \times T(\overline{\mathbb{F}}_p)$ is a constructible subset, we conclude from Lemma 4.3 that $\overline{\sigma}$ is an integral power of the p-th Frobenius automorphism. This completes the proof of Proposition 4.4.

□

Next, we give an alternative proof of [28], Theorem 3.2; [29], Theorem 5.1.3.

Theorem 4.5. *We maintain the notation of Proposition 4.4. Suppose that the integer m is nonnegative (respectively, negative). By a slight abuse of notation, we shall write*

$$F_{X_1}^{(p^m)} : \pi_1^{\mathrm{tame}}(X_1) \xrightarrow{\sim} \pi_1^{\mathrm{tame}}(X_1^{(p^m)}) \quad (\text{respectively, } F_{X_2}^{(p^{-m})} : \pi_1^{\mathrm{tame}}(X_2) \xrightarrow{\sim} \pi_1^{\mathrm{tame}}(X_2^{(p^{-m})}))$$

for the natural $\mathrm{Ker}(\pi_1^{\mathrm{tame}}(X_1^{(p^m)}) \twoheadrightarrow G_k)$-outer (respectively, $\mathrm{Ker}(\pi_1^{\mathrm{tame}}(X_2^{(p^{-m})}) \twoheadrightarrow G_k)$-outer) isomorphism [that lies over the identity automorphism of G_k] induced by $F_{X_1}^{(p^m)}$ (respectively, $F_{X_2}^{(p^{-m})}$). Write

$$\Pi_1^{(p^m)} \quad (\text{respectively, } \Pi_2^{(p^{-m})})$$

for the quotient of $\pi_1^{\mathrm{tame}}(X_1^{(p^m)})$ (respectively, $\pi_1^{\mathrm{tame}}(X_2^{(p^{-m})})$) determined by Π_1 and $F_{X_1}^{(p^m)}$ (respectively, Π_2 and $F_{X_2}^{(p^{-m})}$). Then the natural composite $\mathrm{Ker}(\Pi_2 \twoheadrightarrow G_k)$-outer (respectively, $\mathrm{Ker}(\Pi_2^{(p^{-m})} \twoheadrightarrow G_k)$-outer) isomorphism

$$\Pi_1^{(p^m)} \underset{F_{X_1}^{(p^m)}}{\overset{\sim}{\leftarrow}} \Pi_1 \underset{\sigma}{\overset{\sim}{\rightarrow}} \Pi_2 \quad (\text{respectively, } \Pi_1 \underset{\sigma}{\overset{\sim}{\rightarrow}} \Pi_2 \underset{F_{X_2}^{(p^{-m})}}{\overset{\sim}{\rightarrow}} \Pi_2^{(p^{-m})})$$

arises from a unique isomorphism

$$X_1^{(p^m)} \xrightarrow{\sim} X_2 \quad (\text{respectively, } X_1 \xrightarrow{\sim} X_2^{(p^{-m})})$$

of schemes over k.

Proof. In light of Lemma 4.2 and Proposition 4.4, Theorem 4.5 follows immediately from Corollary 2.14. $\square$

In the remainder of the present section, we prove a certain absolute version of Theorem 4.5. In order to do this, we first prepare some lemmas.

Lemma 4.6. *Suppose that k is a perfect field, and G_k is center-free. Let m be an integer; X a hyperbolic curve over k;*

$$f : X \xrightarrow{\sim} X$$

an automorphism of schemes that lies over the p^m-th Frobenius automorphism of k. Suppose, moreover, that f induces the identity outer automorphism of Π_X. Then $m = 0$, and f is the identity automorphism.

Proof. First, we observe that the natural composite map

$$\mathrm{Aut}_k(X) \longrightarrow \mathrm{Aut}_{G_k}(\Pi_X)/\mathrm{Inn}(\Delta_X) \longrightarrow \mathrm{Out}(\Pi_X)$$

is injective. Indeed, the injectivity of the first arrow is well-known, and the injectivity of the second arrow follows immediately from our assumption that G_k is center-free. In particular, to verify Lemma 4.6, it suffices to verify that $m = 0$. By replacing f by the inverse of f, if necessary, we may assume without loss of generality that m is not positive.

Next, in light of the injectivity of the second arrow in the above display, it follows from our assumption that f induces the identity outer automorphism of Π_X that f determines the identity Δ_X-outer automorphism of Π_X that lies over the identity automorphism of G_k. In particular, f induces the identity automorphism of $\widehat{\mathbb{Z}}^{\Sigma}(1)$. On the other hand, write

$$\phi : X^{(p^{-m})} \xrightarrow{\sim} X$$

for the natural isomorphism of schemes; $g \stackrel{\mathrm{def}}{=} f \circ \phi : X^{(p^{-m})} \xrightarrow{\sim} X$. Note that the automorphism of $\widehat{\mathbb{Z}}^{\Sigma}(1)$ [i.e., the composite $\widehat{\mathbb{Z}}^{\Sigma}(1) \xleftarrow{\sim} \Lambda_{X^{(p^{-m})}} \xrightarrow{\sim} \Lambda_X \xrightarrow{\sim} \widehat{\mathbb{Z}}^{\Sigma}(1)$] induced by ϕ is given by multiplication by p^m. Moreover, since g is an isomorphism over k, it holds that g induces the identity automorphism of $\widehat{\mathbb{Z}}^{\Sigma}(1)$. Therefore, f induces the automorphism of $\widehat{\mathbb{Z}}^{\Sigma}(1)$ given by multiplication by p^{-m}. Thus, we conclude that $1 = p^{-m}$, hence that $m = 0$. This completes the proof of Lemma 4.6. $\square$

Proposition 4.7. *Suppose that k is the perfection of a finitely generated field. Let X be a hyperbolic curve over k; Π a Σ-closed quotient of $\pi_1^{\mathrm{tame}}(X)$. Then the natural quotient $\Pi \twoheadrightarrow G_k$ may be reconstructed, in a purely group-theoretic way, from [the underlying topological group structure of] Π.*

Proof. First, suppose that k is a finite field [or equivalently, Π is topologically finitely generated]. Then Proposition 4.7 follows immediately from a similar argument to the argument applied in the proof of [21], Theorem 2.6, (i) [i.e., an application of the Weil conjecture for abelian varieties over finite fields, together with [21], Proposition A.6, (iv)]. Next, suppose that k is not a finite field [or equivalently, Π is not topologically finitely generated]. Recall that every finitely generated transcendental extension field is Hilbertian [cf. [4], Proposition 13.4.1], and that the operations of forming the perfections of fields do not change their absolute Galois groups. Then it follows from [14], Theorem 2.1, that every topologically finitely generated normal closed subgroup of G_k is trivial. On the other hand, we observe that $\mathrm{Ker}(\pi_1^{\mathrm{tame}}(X) \twoheadrightarrow G_k)$, hence also $\mathrm{Ker}(\Pi \twoheadrightarrow G_k)$, is topologically finitely generated. Thus, we conclude that $\mathrm{Ker}(\Pi \twoheadrightarrow G_k)$ may be characterized as the maximal topologically finitely generated normal closed subgroup of Π. This completes the proof of Proposition 4.7. $\square$

Definition 4.8. Let G_1, G_2 be profinite groups. Then we shall write

$$\mathrm{OutIsom}(G_1, G_2)$$

for the set of outer isomorphisms $G_1 \xrightarrow{\sim} G_2$ of profinite groups.

Here, we record a certain enhancement of a well-known theorem in birational anabelian geometry obtained by Pop.

Theorem 4.9. *Let k_1, k_2 be finitely generated transcendental extension fields of the prime field of characteristic p. Write*

$$\mathrm{Isom}^F(k_2^{\mathrm{pf}}, k_1^{\mathrm{pf}})$$

for the set of isomorphisms $k_2^{\mathrm{pf}} \xrightarrow{\sim} k_1^{\mathrm{pf}}$ of fields considered up to compositions with the Frobenius automorphisms. Then the natural map

$$\mathrm{Isom}^F(k_2^{\mathrm{pf}}, k_1^{\mathrm{pf}}) \longrightarrow \mathrm{OutIsom}(G_{k_1}, G_{k_2})$$

— where we identify G_{k_1}, G_{k_2} with $G_{k_1^{\mathrm{pf}}}$, $G_{k_2^{\mathrm{pf}}}$, respectively — is bijective.

Proof. The injectivity portion of Theorem 4.9 follows immediately from Theorem 3.4, and the surjectivity portion of Theorem 4.9 follows immediately from [24], Theorem 1.3. This completes the proof of Theorem 4.9. $\square$

Finally, we prove an absolute version of Theorem 4.5 as follows.

Theorem 4.10. *Let k_1, k_2 be the perfections of finitely generated transcendental extension fields of the prime field of characteristic p; X_1 a nonisotrivial hyperbolic curve over k_1; X_2 a hyperbolic curve over k_2;*

$$\Pi_1, \ \Pi_2$$

Σ-closed quotients of $\pi_1^{\mathrm{tame}}(X_1)$, $\pi_1^{\mathrm{tame}}(X_2)$, respectively;

$$\sigma : \Pi_1 \xrightarrow{\sim} \Pi_2$$

an outer isomorphism of profinite groups. Then σ arises from a unique isomorphism

$$X_1 \xrightarrow{\sim} X_2$$

of schemes. In particular, the natural maps

$$\mathrm{Isom}(X_1, X_2) \longrightarrow \mathrm{OutIsom}(\pi_1^{\mathrm{tame}}(X_1), \pi_1^{\mathrm{tame}}(X_2)),$$

$$\mathrm{Isom}(X_1, X_2) \longrightarrow \mathrm{OutIsom}(\Pi_{X_1}, \Pi_{X_2})$$

are bijective.

Proof. First, the uniqueness portion follows immediately from Lemma 4.6, together with the injectivity portion of Theorem 4.9. Next, we observe that it follows from Proposition 4.7 that σ induces an outer isomorphism $G_{k_1} \xrightarrow{\sim} G_{k_2}$ of profinite groups. Then, by applying the surjectivity portion of Theorem 4.9, to verify that σ arises from an isomorphism $X_1 \xrightarrow{\sim} X_2$ of schemes, we may assume without loss of generality that $k = k_1 = k_2$, and σ induces the identity outer automorphism of G_k. In this situation, the desired geometricity of σ follows immediately from Theorem 4.5. This completes the proof of Theorem 4.10. $\square$

Remark 4.10.1. The nonisotriviality assumption on X_1 in Theorem 4.10 may not be dropped. Indeed, suppose that k is the perfection of a finitely generated transcendental extension field of the prime field of characteristic p. Write $\phi : G_k \to G_{\mathbb{F}_p}$ for the natural open homomorphism. Let X be an isotrivial hyperbolic curve over k that descends to a hyperbolic curve over $\mathbb{F}_p$; $\sigma \in G_k$ an element such that $\phi(\sigma)$ is nontrivial and does not coincide with any integral power of the p-th Frobenius automorphism [of $\overline{\mathbb{F}}_p$]. Note that since X descends to a hyperbolic curve over $\mathbb{F}_p$, the pro-Σ outer representation

$$\rho : G_k \longrightarrow \mathrm{Out}(\Delta_X)$$

associated to X factors through ϕ. In particular, if we write i_σ for the inner automorphism of G_k determined by σ, then since $G_{\mathbb{F}_p}$ is abelian, it holds that $\rho \circ i_\sigma = \rho$. Thus, in light of the center-freeness of Δ_X, we obtain a Δ_X-outer automorphism of Π_X that induces the trivial outer automorphism on Δ_X and i_σ on G_k. Then the assumption on σ, together with the injectivity portion of Theorem 4.9, immediately implies that [the outer automorphism of Π_X determined by] this Δ_X-outer automorphism does not arise from any automorphism of X.

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